Math 111 – 1

Functions and Graphs

Slope of a line is: $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

- Slope-intercept equation: $y = mx + n$.
- Point-slope equation: $y - y_0 = m(x - x_0)$.
- If two lines are parallel: $m_1 = m_2$.
- If two lines are perpendicular: $m_1 \cdot m_2 = -1$.

Example 1–1: Find the equation of the line with slope $m = 4$, passing through the points $(2, 5)$.

Solution: Using point-slope equation, we find:

$$y - 5 = 4(x - 2) \quad \Rightarrow \quad y = 4x - 3$$

Example 1–2: Find the equation of the line passing through the points $(24, 0)$ and $(8, -6)$.

Solution: The slope is: $m = \dfrac{-6 - 0}{8 - 24} = \dfrac{-6}{-16} = \dfrac{3}{8}$.

Using point-slope equation, we find:

$$y - 0 = \frac{3}{8}(x - 24) \quad \Rightarrow \quad y = \frac{3}{8}x - 9$$

In other words: $3x - 8y = 72$.

Function: A function f defined on a set D of real numbers is a rule that assigns to each number x in D exactly one real number, denoted by $f(x)$.

Intervals:

- Closed interval: $\big[a,\, b\big] = \{x : a \leqslant x \leqslant b\}$
- Open interval: $\big(a,\, b\big) = \{x : a < x < b\}$
- Half-open interval: $\big(a,\, b\big] = \{x : a < x \leqslant b\}$
- Unbounded interval: $\big(a,\, \infty\big) = \{x : a < x\}$

We will use $\mathbb{R}$ to denote all real numbers, in other words the interval $\big(-\infty,\, \infty\big)$.

Domain, Range: The set D of all numbers for which $f(x)$ is defined is called the domain of the function f. The set of all values of $f(x)$ is called the range of f.

Example 1–3: Find the domain and range of the function $f(x) = \sqrt{x-4}$.

Solution: Square root of a negative number is undefined. (In this course, we are not using complex numbers.) Therefore

$$x - 4 \geqslant 0 \quad \Rightarrow \quad x \geqslant 4$$

In other words, domain is: $\big[4,\, \infty\big)$

For $x = 4$ we obtain $y = 0$. We can also obtain any positive value for y. Therefore the range is $\big[0,\, \infty\big)$

Example 1–4: Find the domain and range of the function $f(x) = \dfrac{1}{\sqrt{x-4}}$.

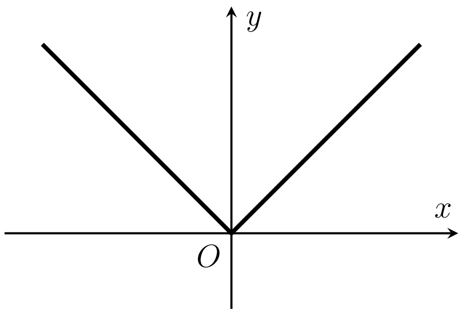
Solution: This is similar to previus exercise, but the function is not defined at $x = 4$. Therefore,

Domain is: $\big(4,\, \infty\big)$

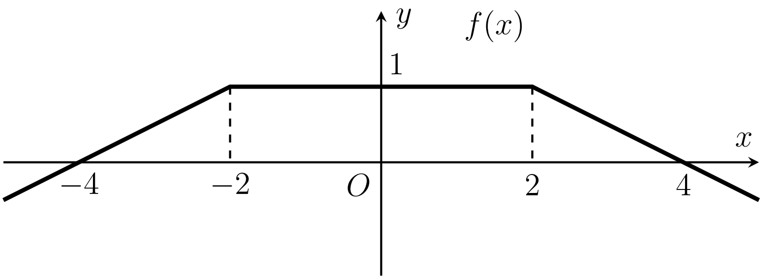
Range is: $\big(0,\, \infty\big)$

Piecewise-Defined Functions: We may define a function using different formulas for different parts of the domain. For example, the absolute value function is:

$$|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$$



Example 1–5: Find the formula of the function $f(x)$:



Solution:

$$f(x) = \begin{cases} \dfrac{x+4}{2} & \text{if} \quad x < -2 \\ 1 & \text{if} \quad -2 \leqslant x \leqslant 2 \\ \dfrac{-x+4}{2} & \text{if} \quad x > 2 \end{cases}$$

Quadratic Equations: The solution of the equation $ax^2 + bx + c = 0$ is:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}, \qquad \Delta = b^2 - 4ac$$

Here, we assume $a \neq 0$.

- If $\Delta > 0$, there are two distinct solutions.
- If $\Delta = 0$, there is a single solution.
- If $\Delta < 0$, there is no real solution.

(In this course, we only consider real numbers)

Example 1–6: Solve $4x^2 + 8x - 5 = 0$.

Solution:
$$\begin{aligned} x &= \frac{-8 \pm \sqrt{64 - 4 \cdot 4 \cdot (-5)}}{2 \cdot 4} \\ &= \frac{-8 \pm \sqrt{144}}{8} \\ &= \frac{-8 + 12}{8} \quad \text{or} \quad \frac{-8 + 12}{8} \end{aligned}$$

$$x_1 = \frac{1}{2}, \quad x_2 = -\frac{5}{2}$$

Example 1–7: Solve $3x^2 + 6x + 4 = 0$.

Solution:
$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= 36 - 48 \\ &= -12 \end{aligned}$$

$\Delta < 0 \quad \Rightarrow \quad$ There is no solution.

EXERCISES

Perform the following operations. Simplify the result:

1–1) $(2^3)^2$

1–2) $\left(\dfrac{1}{16}\right)^{3/4}$

1–3) $72^{1/2}$

1–4) $\sqrt[3]{-125}$

1–5) $\sqrt[3]{\dfrac{8}{1000}}$

1–6) $(a+b)^2$

1–7) $(a+b)(a-b)$

1–8) x^3-1

1–9) $\dfrac{\sqrt{12}}{\sqrt{5}-\sqrt{3}}$

1–10) x^4-100y^4

Find the equations of the following lines:

1–11) Passes through origin and has slope $m = \dfrac{1}{5}$.

1–12) Passes through the point $(-2, 6)$ and has slope $m = 3$.

1–13) Passes through the points $(-8, 2)$ and $(-1, -2)$.

1–14) Passes through $(0, -3)$ and parallel to the line $10y - 5x = 99$.

1–15) Passes through $(9, 12)$ and perpendicular to the line $2x + 5y = 60$.

Find the domain and range of the following functions:

1–16) $f(x) = \sqrt{10 - x}$

1–17) $f(x) = x^2 + 12x + 35$

1–18) $f(x) = 8x - x^2$

1–19) $f(x) = \dfrac{1}{x^2 - 6x + 9}$

1–20) $f(x) = \dfrac{3}{x - 7}$

Solve the following quadratic equations:

1–21) $x^2 - 5x - 24 = 0$

1–22) $2x^2 + 9x - 5 = 0$

1–23) $6x^2 - 7x + 2 = 0$

1–24) $49x^2 - 14x + 1 = 0$

1–25) $4x^2 + 6x + 3 = 0$

1–26) $x^2 - 17x = 0$

1–27) $4x^2 - 20x + 25 = 0$

1–28) $x^2 - 4x + 5 = 0$

1–29) $x^2 - \dfrac{10}{3}x + 1 = 0$

1–30) $x^2 - 2x - 1 = 0$

ANSWERS

1–1) $2^3 \cdot 2^3 = 2^6 = 64$

1–2) $\left(2^{-4}\right)^{3/4} = 2^{-3} = \dfrac{1}{8}$

1–3) $\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$

1–4) $\left[(-5)^3\right]^{1/3} = -5$

1–5) $\dfrac{\sqrt[3]{8}}{\sqrt[3]{1000}} = \dfrac{2}{10} = 0.2$

1–6) $a^2 + 2ab + b^2$

1–7) $a^2 - b^2$

1–8) $(x-1)(x^2+x+1)$

1–9) $\sqrt{15} + 3$

1–10) $(x^2 - 10y^2)(x^2 + 10y^2)$

1–11) $y = \dfrac{1}{5}x$

1–12) $y = 3x + 12$

1–13) $4x + 7y + 18 = 0$

1–14) $x - 2y = 6$

1–15) $5x - 2y = 21$

1–16) Domain: $\big(-\infty, 10\big]$, range: $\big[0, \infty\big)$.

1–17) Domain: $\mathbb{R}$, range: $\big[-1, \infty\big)$.

1–18) Domain: $\mathbb{R}$, range: $\big(-\infty, 16\big)$.

1–19) Domain: $\mathbb{R} \setminus \{3\}$, range: $\big(0, \infty\big)$.

1–20) Domain: $\mathbb{R} \setminus \{7\}$, range: $\big(-\infty, 0\big) \cup \big(0, \infty\big)$.

1–21) $x_1 = 8, \quad x_2 = -3.$

1–22) $x_1 = \dfrac{1}{2}, \quad x_2 = -5.$

1–23) $x_1 = \dfrac{1}{2}, \quad x_2 = \dfrac{2}{3}.$

1–24) $x_1 = \dfrac{1}{7}$. (double root.)

1–25) There is no solution.

1–26) $x_1 = 0, \quad x_2 = 17.$

1–27) $x_1 = \dfrac{5}{2}$. (double root.)

1–28) There is no solution.

1–29) $x_1 = 3, \quad x_2 = \dfrac{1}{3}.$

1–30) $x_1 = 1 + \sqrt{2}, \quad x_2 = 1 - \sqrt{2}.$