Math 111 – 3
Exponential and Logarithmic Functions

Functions of the form

$$f(x) = a^x$$

where a is a positive constant (but $a \neq 1$) are called exponential functions. The domain is:

$$\mathbb{R} = (-\infty, \infty)$$

and the range is

$$(0, \infty)$$

Remember that:

- $a^n = a \cdot a \cdots a$
- $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$
- $a^{1/n} = \sqrt[n]{a}$
- $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

The natural exponential function is:

$$f(x) = e^x$$

where $e = 2.71828\ldots$.

Inverse Functions: If $f\big(g(x)\big) = x$ and $g\big(f(x)\big) = x$, the functions f and g are inverses of each other. For example, the inverse of $f(x) = 2x$ is $g(x) = \dfrac{x}{2}$.

Theorem: A function has an inverse if and only if it is one-to-one and onto.

Logarithmic Functions: The inverse of the exponential function $y = a^x$ is the logarithmic function with base a:

$$y = \log_a x$$

where $a > 0, \quad a \neq 1$.

$$a^{\log_a x} = \log_a(a^x) = x$$

We will use:

- $\log x$ for $\log_{10} x$ (common logarithm)
- $\ln x$ for $\log_e x$ (natural logarithm)

We can easily see that,

$$a^x \cdot a^y = a^{x+y} \quad \Rightarrow \quad \log_a(AB) = \log_a A + \log_a B$$

As a result of this,

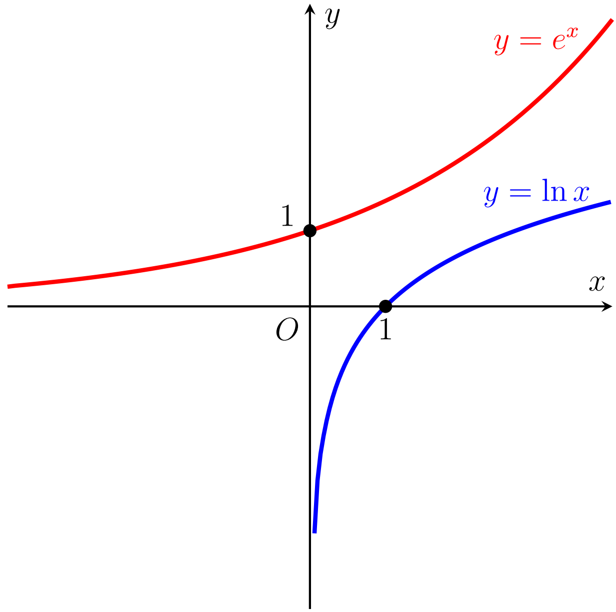
- $\log_a \left(\dfrac{A}{B}\right) = \log_a A - \log_a B$
- $\log_a \left(\dfrac{1}{B}\right) = -\log_a B$
- $\log_a (A^r) = r \log_a A$

Any logarithm can be expressed in terms of the natural logarithm:

$$\log_a(x) = \frac{\ln x}{\ln a}$$

Any exponential can be expressed in terms of the natural exponential:

$$a^x = e^{x \ln a}$$



y
$y = e^x$
$y = \ln x$
1
x
O
1

EXERCISES

Simplify the following:

3–1) $\log 400$

3–2) $\log 288$

3–3) $\log_9 27$

3–4) $\log_8 16$

3–5) $\log_2 1250$

3–6) $\log_3 \dfrac{\sqrt{3}}{81}$

3–7) $e^{2x+5\ln x}$

3–8) $\ln \dfrac{e}{\sqrt[3]{e}}$

3–9) $2^{3x+4\log_2 x}$

3–10) $3^{2\log_9 x}$

3–11) $5^{\log_{25} x}$

3–12) $10^{1+\log(2x)}$

Solve the following equations.

3–13) $5 = \left(5\sqrt{5}\right)^x$

3–14) $\log_x 12 = \dfrac{1}{2}$

3–15) $\log_x 77 = -1$

3–16) $\log_x 2 = 3$

3–17) $\log_x 64 = 4$

3–18) $\log_3 x = 5$

3–19) $\log_9(18x) = 2$

3–20) $\log_5 x = -\dfrac{1}{2}$

3–21) $\log(\log x) = 0$

3–22) $\ln(\ln x) = 1$

3–23) $2^x = 100$

3–24) $2^{4x+4} = 8^{x-1}$

ANSWERS

3–1) $2 + 2\log 2$

3–2) $2\log 3 + 5\log 2$

3–3) $\frac{3}{2}$

3–4) $\frac{4}{3}$

3–5) $1 + 4\log_2 5$

3–6) $-\frac{7}{2}$

3–7) $x^5 e^{2x}$

3–8) $\frac{2}{3}$

3–9) $x^4 8^x$

3–10) x

3–11) $\sqrt{x}$

3–12) $20x$

3–13) $x = \dfrac{2}{3}$

3–14) $x = 144$

3–15) $x = \dfrac{1}{77}$

3–16) $x = 2^{1/3}$

3–17) $x = 2\sqrt{2}$

3–18) $x = 243$

3–19) $x = \dfrac{9}{2}$

3–20) $x = \dfrac{1}{\sqrt{5}}$

3–21) $x = 10$

3–22) $x = e^e$

3–23) $x = \dfrac{2}{\log 2}$

3–24) $x = -7$