

# Math 111 – 4

## Limits

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**Definition:** We say that  $f(x)$  has the limit  $L$  at  $x = a$  if  $f(x)$  gets as close to  $L$  as we like, when  $x$  approaches  $a$ . (without getting equal to  $a$ ) We write this as:

$$\lim_{x \rightarrow a} f(x) = L$$

**Limit Laws:** If both of the limits  $\lim_{x \rightarrow a} f(x) = L$  and  $\lim_{x \rightarrow a} g(x) = M$  exist, then:

- $\lim_{x \rightarrow a} f \pm g = L \pm M$
- $\lim_{x \rightarrow a} fg = LM$
- $\lim_{x \rightarrow a} \frac{f}{g} = \frac{L}{M}$  (if  $M \neq 0$ )
- $\lim_{x \rightarrow a} \sqrt[n]{f} = \sqrt[n]{L}$
- $\lim_{x \rightarrow a} f(g(x)) = f(M)$   
(If  $f$  is continuous at  $M$ )

**Example 4–1:** Evaluate the limit  $\lim_{x \rightarrow 2} \frac{3}{x - 2}$  (if it exists):

**Solution:** As  $x$  approaches 2, the function  $\frac{3}{x-2}$  gets larger and larger without any bound. Therefore the limit does not exist. (Limit DNE.)

**Example 4–2:** Evaluate the following limit (if it exists):

$$\lim_{x \rightarrow -5} \frac{x^2 + 6x + 5}{x^2 + 7x + 10}$$

$$\begin{aligned}\mathbf{Solution:} \quad \lim_{x \rightarrow -5} \frac{x^2 + 6x + 5}{x^2 + 7x + 10} &= \lim_{x \rightarrow -5} \frac{(x+5)(x+1)}{(x+5)(x+2)} \\&= \lim_{x \rightarrow -5} \frac{(x+1)}{(x+2)} \\&= \frac{4}{3}\end{aligned}$$

**Example 4–3:** Evaluate the following limit (if it exists):

$$\lim_{x \rightarrow 0} \frac{\sqrt{9 + 12x} - 3}{x}$$

**Solution:** Multiply both numerator and denominator by the conjugate of the numerator:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{9 + 12x} - 3}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{9 + 12x} - 3}{x} \cdot \frac{\sqrt{9 + 12x} + 3}{\sqrt{9 + 12x} + 3} \\&= \lim_{x \rightarrow 0} \frac{9 + 12x - 9}{x(\sqrt{9 + 12x} + 3)} \\&= \lim_{x \rightarrow 0} \frac{12}{\sqrt{9 + 12x} + 3} \\&= 2\end{aligned}$$

## EXERCISES

Evaluate the following limits, if they exist:

$$4-1) \lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$$

$$4-2) \lim_{x \rightarrow -4} \frac{x^2 + 11x + 28}{x^2 + 12x + 32}$$

$$4-3) \lim_{x \rightarrow 5} \frac{x^2 - 25}{x^2 - 10x + 25}$$

$$4-4) \lim_{x \rightarrow 1} \frac{x^3 - x^2 - x + 1}{x^2 - x}$$

$$4-5) \lim_{x \rightarrow 3} \frac{x^4 - 81}{x^3 - 27}$$

$$4-6) \lim_{x \rightarrow 1} \frac{x^n - 1}{x - 1}$$

$$4-7) \lim_{x \rightarrow 0} \frac{\sqrt{x+16} - 4}{x}$$

$$4-8) \lim_{x \rightarrow 3} \frac{x - 3}{\sqrt{x+6} - 3}$$

$$4-9) \lim_{x \rightarrow 64} \frac{\sqrt{x} - 8}{x - 64}$$

$$4-10) \lim_{x \rightarrow 0} \frac{\sqrt{2x+1} - 3}{x}$$

$$4-11) \lim_{x \rightarrow 7} \frac{\sqrt{4x+8} - 6}{x - 7}$$

Evaluate the following limits, if they exist:

$$4-12) \lim_{x \rightarrow \infty} \frac{x(x^2 - 5x + 14)}{7 - 4x^3}$$

$$4-13) \lim_{x \rightarrow \infty} \frac{3x^2 + 12x + 9}{(x^2 - 1)(x^2 + 1)}$$

$$4-14) \lim_{x \rightarrow \infty} \frac{2 + 3x - 4x^4}{\sqrt{x}(1 - 17x + 8x^3)}$$

$$4-15) \lim_{x \rightarrow \infty} \sqrt{x^2 + 6x} - x$$

$$4-16) \lim_{x \rightarrow \infty} \frac{1}{\ln(x^2)}$$

$$4-17) \lim_{x \rightarrow \infty} \frac{8e^x}{4 + 5e^x}$$

$$4-18) \lim_{x \rightarrow \infty} \sqrt{x^2 + 4x} - \sqrt{x^2 - 10x + 1}$$

$$4-19) \lim_{x \rightarrow \infty} \frac{x^4 - 16}{(2x - 1)(2x + 1)(x^2 + 1)}$$

$$4-20) \lim_{x \rightarrow \infty} \sqrt{x^2 - 12x + 24} - \sqrt{x^2 + 10x + 5}$$

$$4-21) \lim_{x \rightarrow \infty} \frac{1}{2x - \sqrt{4x^2 - 5x + 6}}$$

$$4-22) \lim_{x \rightarrow \infty} \sqrt{2x^2 - 1} - \sqrt{x^2 + 1}$$

# ANSWERS

**4–1)** 12

**4–2)**  $\frac{3}{4}$

**4–3)** Limit DNE.

**4–4)** 0

**4–5)** 4

**4–6)**  $n$

**4–7)**  $\frac{1}{8}$

**4–8)** 6

**4–9)**  $\frac{1}{16}$

**4–10)** Limit DNE.

**4–11)**  $\frac{1}{3}$

$$4-12) -\frac{1}{4}$$

$$4-13) 0$$

$$4-14) -\infty$$

$$4-15) 3$$

$$4-16) 0$$

$$4-17) \frac{8}{5}$$

$$4-18) 7$$

$$4-19) \frac{1}{4}$$

$$4-20) -11$$

$$4-21) \frac{4}{5}$$

$$4-22) \infty$$