

# Math 111 – 5

## One Sided Limits, Continuity

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### 5.1 One Sided Limits

If  $x$  approaches  $a$  from right, taking values larger than  $a$  only, we denote this by  $x \rightarrow a^+$ . If  $f(x)$  approaches  $L$  as  $x \rightarrow a^+$ , then we say that  $L$  is the right-hand limit of  $f$  at  $a$ .

$$\lim_{x \rightarrow a^+} f(x) = L$$

We define the left-hand limit of  $f$  at  $a$  similarly:

$$\lim_{x \rightarrow a^-} f(x) = L$$

**Theorem:** The limit  $\lim_{x \rightarrow a} f(x) = L$  exists if and only if both one sided limits

$$\lim_{x \rightarrow a^+} f(x) \quad \text{and} \quad \lim_{x \rightarrow a^-} f(x)$$

exist and are equal to  $L$ .

**Example 5–1:** Find the limit  $\lim_{x \rightarrow 0^+} \ln x$  if it exists.

**Solution:** If you remember the graph of  $f(x) = \ln x$  you will see that:

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

Note that the question  $\lim_{x \rightarrow 0} \ln x$  would be meaningless.

**Example 5–2:** Let  $f(x) = \begin{cases} 2x - 1 & \text{if } x < 1 \\ 5x - 2 & \text{if } x > 1 \end{cases}$

Find the limits  $\lim_{x \rightarrow 1^-} f(x)$ ,  $\lim_{x \rightarrow 1^+} f(x)$  and  $\lim_{x \rightarrow 1} f(x)$ .

**Solution:**  $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 2x - 1 = 1$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 5x - 2 = 3$$

$$\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x) \text{ therefore } \lim_{x \rightarrow 1} f(x)$$

does not exist.

**Example 5–3:** Let  $f(x) = \begin{cases} 2 - x^2 & \text{if } x < 0 \\ 7 & \text{if } x = 0 \\ e^x + e^{-x} & \text{if } x > 0 \end{cases}$

Find the limits  $\lim_{x \rightarrow 0^-} f(x)$ ,  $\lim_{x \rightarrow 0^+} f(x)$  and  $\lim_{x \rightarrow 0} f(x)$ .

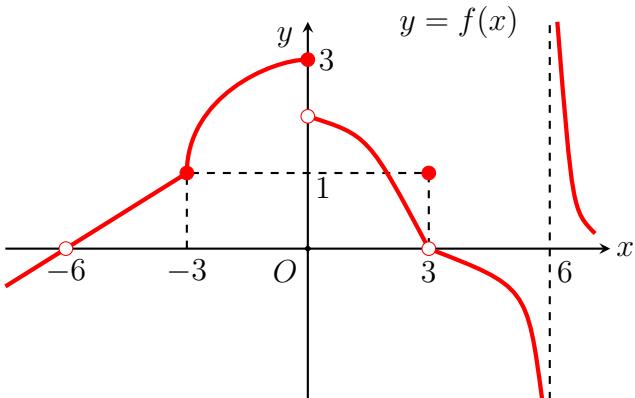
**Solution:**  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 2 - x^2 = 2$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^x + e^{-x} = 2$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = 2 \text{ therefore } \lim_{x \rightarrow 0} f(x) = 2.$$

(Note that the function value  $f(0) = 7$  does not have any effect on the limit.)

**Example 5–4:** Find the limits based on the function  $f(x)$  in the figure: (If they exist.)



a)  $\lim_{x \rightarrow -6^-} f(x)$ ,  $\lim_{x \rightarrow -6^+} f(x)$ ,  $\lim_{x \rightarrow -6} f(x)$ .

b)  $\lim_{x \rightarrow -3^-} f(x)$ ,  $\lim_{x \rightarrow -3^+} f(x)$ ,  $\lim_{x \rightarrow -3} f(x)$ .

c)  $\lim_{x \rightarrow 0^-} f(x)$ ,  $\lim_{x \rightarrow 0^+} f(x)$ ,  $\lim_{x \rightarrow 0} f(x)$ .

d)  $\lim_{x \rightarrow 3^-} f(x)$ ,  $\lim_{x \rightarrow 3^+} f(x)$ ,  $\lim_{x \rightarrow 3} f(x)$ .

e)  $\lim_{x \rightarrow 6^-} f(x)$ ,  $\lim_{x \rightarrow 6^+} f(x)$ ,  $\lim_{x \rightarrow 6} f(x)$ .

**Solution:** a) 0, 0, 0.

b) 1, 1, 1.

c) 3, 2, does not exist.

d) 0, 0, 0.

e)  $-\infty$ ,  $\infty$ , does not exist.

**Example 5–5:** Evaluate the limit (if it exists)

$$\lim_{x \rightarrow 8^+} \frac{x^2 - 10x + 16}{\sqrt{x - 8}}$$

**Solution:** Note that square root of a negative number is not defined, so  $x$  should not take values less than 8.

Therefore the question

$$\lim_{x \rightarrow 8} \frac{x^2 - 10x + 16}{\sqrt{x - 8}}$$

would be meaningless.

Now if we factor  $x^2 - 10x + 16$  as:

$$\begin{aligned} x^2 - 10x + 16 &= (x - 8)(x - 2) \\ &= \sqrt{x - 8} \sqrt{x - 8} (x - 2) \end{aligned}$$

we obtain:

$$\begin{aligned} \lim_{x \rightarrow 8^+} \frac{x^2 - 10x + 16}{\sqrt{x - 8}} &= \lim_{x \rightarrow 8^+} \frac{\sqrt{x - 8} \sqrt{x - 8} (x - 2)}{\sqrt{x - 8}} \\ &= \lim_{x \rightarrow 8^+} \sqrt{x - 8} (x - 2) \\ &= 0 \end{aligned}$$

## 5.2 Continuity

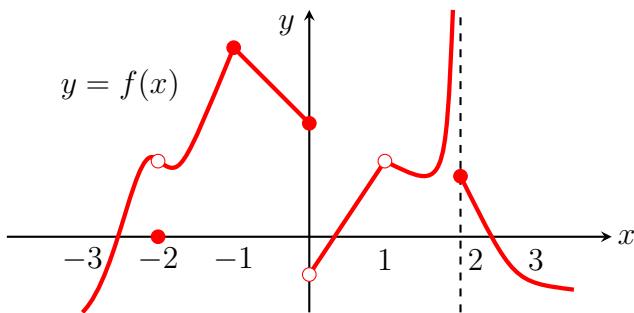
We say that  $f$  is continuous at  $a$  if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

In other words:

- $f$  must be defined at  $a$ .
- $\lim_{x \rightarrow a} f(x)$  must exist.
- The limit must be equal to the function value.

**Example 5–6:** Determine the points where  $f(x)$  is discontinuous:



**Solution:**  $f(x)$  is discontinuous at:

- $x = -2$ , limit and function value are different.
- $x = 0$ , limit does not exist.
- $x = 1$ , function is undefined.
- $x = 2$ , limit does not exist.

$$\text{Example 5-7: Let } f(x) = \begin{cases} 2x^2 + a & \text{if } x < 2 \\ b & \text{if } x = 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$$

Find  $a$  and  $b$  if  $f(x)$  is continuous at  $x = 2$ .

$$\text{Solution: } \lim_{x \rightarrow 2^-} f(x) = 8 + a \text{ and } \lim_{x \rightarrow 2^+} f(x) = 4.$$

If  $f$  is continuous at  $x = 2$ , then

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\Rightarrow 8 + a = b = 4$$

We find  $a = -4$ ,  $b = 4$ .

$$\text{Example 5-8: Let } f(x) = \begin{cases} \log\left(\frac{x}{2} + b\right) & \text{if } x < 8 \\ x\left(\sqrt{x-8} + \frac{1}{4}\right) & \text{if } x \geq 8 \end{cases}$$

Find  $b$  if  $f(x)$  is continuous at  $x = 8$ .

$$\text{Solution: } \lim_{x \rightarrow 8^+} f(x) = 2$$

$$\lim_{x \rightarrow 8^-} f(x) = \log(4 + b)$$

If  $f$  is continuous, these limits must be equal.

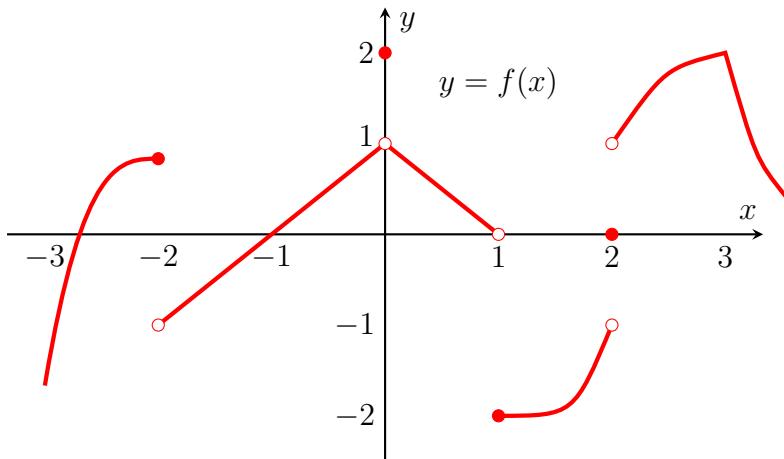
$$\log(4 + b) = 2$$

$$4 + b = 100$$

$$b = 96$$

## EXERCISES

5–1) Find the limits based on the figure:



a)  $\lim_{x \rightarrow -2^-} f(x)$ ,  $\lim_{x \rightarrow -2^+} f(x)$ ,  $\lim_{x \rightarrow -2} f(x)$ .

b)  $\lim_{x \rightarrow -1^-} f(x)$ ,  $\lim_{x \rightarrow -1^+} f(x)$ ,  $\lim_{x \rightarrow -1} f(x)$ .

c)  $\lim_{x \rightarrow 0^-} f(x)$ ,  $\lim_{x \rightarrow 0^+} f(x)$ ,  $\lim_{x \rightarrow 0} f(x)$ .

d)  $\lim_{x \rightarrow 1^-} f(x)$ ,  $\lim_{x \rightarrow 1^+} f(x)$ ,  $\lim_{x \rightarrow 1} f(x)$ .

e)  $\lim_{x \rightarrow 2^-} f(x)$ ,  $\lim_{x \rightarrow 2^+} f(x)$ ,  $\lim_{x \rightarrow 2} f(x)$ .

f)  $\lim_{x \rightarrow 3^-} f(x)$ ,  $\lim_{x \rightarrow 3^+} f(x)$ ,  $\lim_{x \rightarrow 3} f(x)$ .

5–2) Find the points where  $f(x)$  of previous question is discontinuous.

Evaluate the following limits: (If they exist)

$$5-3) \lim_{x \rightarrow 7^-} \frac{2}{x - 7}$$

$$5-4) \lim_{x \rightarrow 7^+} \frac{2}{x - 7}$$

$$5-5) \lim_{x \rightarrow 7^-} \frac{|x - 7|}{x - 7}$$

$$5-6) \lim_{x \rightarrow 7^+} \frac{|x - 7|}{x - 7}$$

$$5-7) \lim_{x \rightarrow 3^+} \sqrt{\frac{x - 3}{x + 3}}$$

$$5-8) \lim_{x \rightarrow 0^+} \frac{\sqrt{16 + 3x} - 4}{x}$$

$$5-9) \lim_{x \rightarrow -2^+} \frac{|x^2 - 4|}{x + 2}$$

$$5-10) \lim_{x \rightarrow -2^-} \frac{|x^2 - 4|}{x + 2}$$

$$5-11) \lim_{x \rightarrow 0^+} \frac{2x^2 + 3x|x|}{x|x|}$$

$$5-12) \lim_{x \rightarrow 0^-} \frac{2x^2 + 3x|x|}{x|x|}$$

Find all the discontinuities of the following functions:

$$\mathbf{5-13)} \quad f(x) = \frac{x^2 - 2}{x^2 - 4}$$

$$\mathbf{5-14)} \quad f(x) = \frac{|x - a|}{x - a}$$

$$\mathbf{5-15)} \quad f(x) = \frac{x^2 - 5x + 6}{x^2 - 4x + 3}$$

$$\mathbf{5-16)} \quad f(x) = \frac{1}{e^{2x} - e^{3x}}$$

$$\mathbf{5-17)} \quad f(x) = \frac{x - 5}{x^2 - 25}$$

$$\mathbf{5-18)} \quad f(x) = \frac{1}{1 - |x|}$$

$$\mathbf{5-19)} \quad f(x) = \begin{cases} -1 + x & \text{if } x \leq 0 \\ 1 + x^2 & \text{if } x > 0 \end{cases}$$

$$\mathbf{5-20)} \quad f(x) = \begin{cases} 12x - 20 & \text{if } x < 2 \\ 8 & \text{if } x = 2 \\ x^2 & \text{if } x > 2 \end{cases}$$

Find the values of constants that will make the following functions continuous everywhere:

$$5-21) \quad f(x) = \begin{cases} a + bx^2 & \text{if } x < 0 \\ b & \text{if } x = 0 \\ 2 + e^{-x} & \text{if } x > 0 \end{cases}$$

$$5-22) \quad f(x) = \begin{cases} cx^2 - 2 & \text{if } x \leq 2 \\ \frac{x}{c} & \text{if } x > 2 \end{cases}$$

$$5-23) \quad f(x) = \begin{cases} x^2 - c^2 & \text{if } x \leq 1 \\ (x - c)^2 & \text{if } x > 1 \end{cases}$$

$$5-24) \quad f(x) = \begin{cases} e^{ax} & \text{if } x \leq 0 \\ \ln(b + x^2) & \text{if } x > 0 \end{cases}$$

# ANSWERS

**5–1)**

a) 1, -1, Does Not Exist.

b) 0, 0, 0.

c) 1, 1, 1.

d) 0, -2, DNE.

e) -1, 1, DNE.

f) 2, 2, 2.

**5–2)**

$x = -2$ .

$x = 0$ .

$x = 1$ .

$x = 2$ .

**5–3)**  $-\infty$

**5–4)**  $\infty$

**5–5)** -1

**5–6)** 1

**5–7)** 0

**5–8)**  $\frac{3}{8}$

**5–9)** 4

**5–10)** -4

**5–11)** 5

**5–12)** 1

**5–13)**  $x = 2$  and  $x = -2$ .

**5–14)**  $x = a$ .

**5–15)**  $x = 1, x = 3$ .

**5–16)**  $x = 0$ .

**5–17)**  $x = -5, x = 5$ .

**5–18)**  $x = 1$  and  $x = -1$ .

**5–19)**  $x = 0$ .

**5–20)**  $x = 2$ .

**5–21)**  $a = b = 3$

**5–22)**  $c = 1$ , or  $c = -\frac{1}{2}$

**5–23)**  $c = 0$ , or  $c = 1$

**5–24)**  $b = e$ ,  $a$  is arbitrary.