

# Math 111 – 6

## Derivatives, Chain Rule

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**Definition and Notation:** The derivative of the function  $f(x)$  is the function  $f'(x)$  defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Or, equivalently:  $f'(x) = \lim_{a \rightarrow x} \frac{f(x) - f(a)}{x - a}$

We can think of the derivative as

- The rate of change of a function  $f$ , or
- The slope of the curve of  $y = f(x)$ .

We will use  $y'$ ,  $f'(x)$ ,  $\frac{dy}{dx}$ ,  $\frac{d}{dx}f(x)$  to denote derivatives and  $f'(a)$ ,  $\left. \frac{dy}{dx} \right|_{x=a}$  to denote their values at a certain point.

Note that derivative is a function, its value at a point is a number.

**Higher Order Derivatives:** We can find the derivative of the derivative of a function. It is called second derivative and denoted by:

$$y'', \quad f''(x), \quad \frac{d^2y}{dx^2}.$$

For third derivative, we use  $f'''(x)$  but for fourth and higher derivatives, we use the notation  $f^{(4)}(x)$ ,  $f^{(5)}(x)$  etc.

**Example 6–1:** Let  $f(x) = 7x^3 - 18x$ . Find  $f'(x)$ ,  $f''(x)$ ,  $f'''(x)$  and  $f^{(4)}(x)$ .

**Solution:**  $f'(x) = 21x^2 - 18$

$$f''(x) = 42x$$

$$f'''(x) = 42$$

$$f^{(4)}(x) = 0$$

**Differentiation Formulas:** Using the definition of derivative, we obtain:

- Derivative of a constant is zero, i.e.  $\frac{dc}{dx} = 0$

- Derivative of  $f(x) = x$  is 1:

$$\frac{d}{dx} x = 1$$

- Derivative of  $f(x) = x^n$  is:

$$\frac{d}{dx} x^n = nx^{n-1}$$

- Derivative of  $f(x) = \sqrt{x}$  is:

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$

- If  $f$  is a function and  $c$  is a constant, then

$$(cf)' = cf'$$

- If  $f$  and  $g$  are functions, then

$$(f + g)' = f' + g'$$

**Example 6–2:** Evaluate the derivative of  $f(x) = \frac{7x^3 - 18x}{x}$ .

**Solution:** First we have to simplify:

$$f(x) = 7x^2 - 18$$

Then we use the differentiation rules:

$$f'(x) = 14x$$

**Example 6–3:** Find the equation of the tangent line to the graph of  $f(x) = x^2$  at the point  $(1, 1)$ .

**Solution:**  $f'(x) = 2x \Rightarrow m = f'(1) = 2$

Using point slope equation  $\left( y - y_0 = m(x - x_0) \right)$  we find the equation of the tangent line as:

$$(y - 1) = 2(x - 1) \Rightarrow y = 2x - 1$$

### Differentiation Rules:

**Product Rule:** If  $f$  and  $g$  are differentiable at  $x$ , then  $fg$  is differentiable at  $x$  and

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

or more briefly:

$$(fg)' = f'g + fg'$$

**Example 6–4:** Find the derivative of  $f(x) = (x^4 + 14x)(7x^3 + 17)$

**Solution:**  $f'(x) = (4x^3 + 14)(7x^3 + 17) + (x^4 + 14x)21x^2$

**Reciprocal Rule:** If  $f$  is differentiable at  $x$  and if  $f(x) \neq 0$  then:

$$\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}$$

**Example 6–5:** Using the reciprocal rule, find the derivative of  $f(x) = \frac{1}{x^n}$ .

**Solution:**  $f'(x) = \frac{-nx^{n-1}}{x^{2n}} = -\frac{n}{x^{n+1}} = -n x^{-n-1}$

**Example 6–6:** Find the derivative of  $f(x) = \frac{1}{8x^2 + 12x + 1}$ .

**Solution:**  $f'(x) = -\frac{16x + 12}{(8x^2 + 12x + 1)^2}$

**Quotient Rule:** If  $f$  and  $g$  are differentiable at  $x$ , and  $g(x) \neq 0$

then  $\frac{f}{g}$  is differentiable at  $x$ :

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

**Example 6–7:** Find the derivative of  $f(x) = \frac{2x + 3}{5x^2 + 7}$ .

**Solution:** 
$$\begin{aligned} f'(x) &= \frac{2(5x^2 + 7) - 10x(2x + 3)}{(5x^2 + 7)^2} \\ &= \frac{-10x^2 - 30x + 14}{(5x^2 + 7)^2} \end{aligned}$$

## Exponentials and Logarithms:

$$\frac{d}{dx} e^x = e^x$$

$e^x$  is the only nonzero function whose derivative is itself.

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

**Chain Rule:** If  $f$  and  $g$  are differentiable then  $f(g(x))$  is also differentiable and

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

or more briefly

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

**Example 6–8:** Find  $\frac{d}{dx}(3x^2 + 1)^5$ .

**Solution:** Here  $u = 3x^2 + 1$  and  $y = u^5$ . Using the above formula, we obtain:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\&= 5u^4 \cdot 6x \\&= 5(3x^2 + 1)^4 \cdot 6x \\&= 30x(3x^2 + 1)^4\end{aligned}$$

**Example 6–9:** Find  $('x)f$  where  $f(x) = e^{x^5}$ .

**Solution:** Here  $u = x^5$  and  $f = e^u$ . Using the formula, we obtain:

$$f'(x) = e^{x^5} \cdot 5x^4$$

## Logarithmic Differentiation:

Logarithm transforms products into sums. This helps in finding derivatives of some complicated functions.

For example if

$$y = \frac{(x^3 + 1)(x^2 - 1)}{x^8 + 6x^4 + 1}$$

then

$$\ln y = \ln(x^3 + 1) + \ln(x^2 - 1) - \ln(x^8 + 6x^4 + 1)$$

Derivative of both sides gives:

$$\frac{y'}{y} = \frac{3x^2}{x^3 + 1} + \frac{2x}{x^2 - 1} - \frac{8x^7 + 24x^3}{x^8 + 6x^4 + 1}$$

**Example 6–10:** Find the derivative of the function:

$$y = f(x) = (x + e^x)^{\ln x}$$

**Solution:**  $\ln y = \ln x \ln(x + e^x)$

$$(\ln y)' = \frac{1}{x} \ln(x + e^x) + \frac{1 + e^x}{x + e^x} \ln x$$

$$\frac{y'}{y} = \frac{\ln(x + e^x)}{x} + \frac{1 + e^x}{x + e^x} \ln x$$

$$y' = (x + e^x)^{\ln x} \left( \frac{\ln(x + e^x)}{x} + \frac{1 + e^x}{x + e^x} \ln x \right)$$

## EXERCISES

Evaluate the derivatives of the following functions:

$$\mathbf{6-1)} \quad f(x) = 1 - \sqrt{x}$$

$$\mathbf{6-2)} \quad f(x) = 4 + 3x - 12x^3$$

$$\mathbf{6-3)} \quad f(x) = x^{-1} + 4x^{-2}$$

$$\mathbf{6-4)} \quad f(x) = \frac{1}{\sqrt[4]{x}}$$

$$\mathbf{6-5)} \quad f(x) = \frac{1}{x} - \frac{2}{x^2}$$

$$\mathbf{6-6)} \quad f(x) = 20x^{-4} + 4x^{1/4}$$

$$\mathbf{6-7)} \quad f(x) = \frac{x^3 - x}{\sqrt{x}}$$

$$\mathbf{6-8)} \quad f(x) = \frac{2x^4 - x^3 - 1}{x}$$

$$\mathbf{6-9)} \quad f(x) = (x^2 + 2)(x^2 - 3)$$

$$\mathbf{6-10)} \quad f(x) = \frac{x}{x^2 + 4}$$

$$\mathbf{6-11)} \quad f(x) = \frac{x^2 + 12}{5x - 2}$$

$$\mathbf{6-12)} \quad f(x) = \frac{x^{3/2} + x^{-1/2}}{\sqrt{x} + \frac{1}{\sqrt{x}}}$$

Evaluate the derivatives of the following functions:

$$\mathbf{6-13)} \quad f(x) = x^{12}e^x$$

$$\mathbf{6-14)} \quad f(x) = x^2 \ln(x^3)$$

$$\mathbf{6-15)} \quad f(x) = \frac{5x}{\ln x}$$

$$\mathbf{6-16)} \quad f(x) = \frac{e^x}{1+x^2}$$

$$\mathbf{6-17)} \quad f(x) = \frac{1}{1+2x+3e^x}$$

$$\mathbf{6-18)} \quad f(x) = \frac{1}{e^x + 2 \ln x}$$

$$\mathbf{6-19)} \quad f(x) = x^4 e^x \ln x$$

$$\mathbf{6-20)} \quad f(x) = (x + e^x)(x^2 + \ln x)$$

$$\mathbf{6-21)} \quad f(x) = \frac{4x^2 - 5x}{2e^x - 3x}$$

$$\mathbf{6-22)} \quad f(x) = \frac{1}{\ln(4x)}$$

$$\mathbf{6-23)} \quad f(x) = e^x e^x e^x$$

$$\mathbf{6-24)} \quad f(x) = \frac{2 - 3 \ln x}{5 \ln x + 1}$$

Evaluate the derivatives of the following functions using chain rule:

$$\mathbf{6-25)} \quad f(x) = (1 + x^4)^2$$

$$\mathbf{6-26)} \quad f(x) = e^{x^3}$$

$$\mathbf{6-27)} \quad f(x) = \ln(1 + x^2)$$

$$\mathbf{6-28)} \quad f(x) = (5 + x + 2x^3)^7$$

$$\mathbf{6-29)} \quad f(x) = \frac{x}{\sqrt{3x^2 + 2}}$$

$$\mathbf{6-30)} \quad f(x) = \frac{1}{(x^2 - 4x)^3}$$

$$\mathbf{6-31)} \quad f(x) = \left(\frac{2x}{x-1}\right)^5$$

$$\mathbf{6-32)} \quad f(x) = (e^{3x} + 1)^5$$

$$\mathbf{6-33)} \quad f(x) = \sqrt{1 + \ln x}$$

$$\mathbf{6-34)} \quad f(x) = \sqrt{x^2 + 2e^{3x}}$$

$$\mathbf{6-35)} \quad f(x) = 4^{x^2+5x}$$

$$\mathbf{6-36)} \quad f(x) = xe^x \log_3(x + x^4)$$

Find  $f''$ :

**6–37)**  $f(x) = 5^{2x}$

**6–38)**  $f(x) = \ln(3x)$

**6–39)**  $f(x) = \sqrt{2+x}$

**6–40)**  $f(x) = x^7 e^{-x}$

Find  $f'$  using logarithmic differentiation:

**6–41)**  $f(x) = (1+2x)^7(x^3+1)^4$

**6–42)**  $f(x) = \frac{(3x^4+x^2)^6}{(1+x+x^2)^8}$

**6–43)**  $f(x) = (\ln x)^x$

Find the equation of the line tangent to  $f(x)$  at  $x_0$ :

**6–44)**  $f(x) = 2x^2 - 8x + 4, \quad x_0 = 2.$

**6–45)**  $f(x) = x\sqrt{2x+4}, \quad x = 0.$

**6–46)**  $f(x) = x^2(1-x)^2, \quad x = 2.$

**6–47)**  $f(x) = \frac{1}{1+x^2}, \quad x = 0.$

Evaluate the derivatives of the following functions at the point  $x = a$ . In other words, find the value of  $f'(a)$ .

**6–48)**  $f(x) = \frac{2x^2 - 3x + 12}{x}, \quad a = 2.$

**6–49)**  $f(x) = x^{3/5}, \quad a = 32.$

**6–50)**  $f(x) = \frac{5 + 3x^2}{8 + 4x}, \quad a = 0.$

**6–51)**  $f(x) = \frac{\ln x}{x^4}, \quad a = 1.$

**6–52)**  $f(x) = (1 + 2x)e^x, \quad a = 0.$

**6–53)**  $f(x) = \sqrt{10 - e^{-x}}, \quad a = 0.$

**6–54)**  $f(x) = \ln\left(\frac{x - 2}{3x - 3}\right), \quad a = 5.$

**6–55)**  $f(x) = \left(2x + \frac{3}{x}\right)^2, \quad a = \frac{1}{2}.$

**6–56)**  $f(x) = (4x + e^{5x})^3, \quad a = 0.$

**6–57)**  $f(x) = \frac{1}{2 + 4x + 8e^{2x}}, \quad a = 0.$

**6–58)**  $f(x) = x \ln \sqrt{1 + 2x}, \quad a = 1.$

**6–59)**  $f(x) = x^2 e^{1/x}, \quad a = \frac{1}{4}.$

**6–60)**  $f(x) = \ln\left(\frac{xe^x}{1 + x^2}\right), \quad a = 2.$

## ANSWERS

**6–1)**  $f'(x) = \frac{-1}{2\sqrt{x}}$

**6–2)**  $f'(x) = 3 - 36x^2$

**6–3)**  $f'(x) = -x^{-2} - 8x^{-3}$

**6–4)**  $f'(x) = -\frac{1}{4}x^{-5/4}$

**6–5)**  $f'(x) = -\frac{1}{x^2} + \frac{4}{x^3}$

**6–6)**  $f'(x) = -80x^{-5} + x^{-3/4}$

**6–7)**  $f'(x) = \frac{5}{2}x^{3/2} - \frac{1}{2}x^{-1/2}$

**6–8)**  $f'(x) = 6x^2 - 2x + \frac{1}{x^2}$

**6–9)**  $f'(x) = 4x^3 - 2x$

**6–10)**  $f'(x) = \frac{4 - x^2}{(x^2 + 4)^2}$

**6–11)**  $f'(x) = \frac{5x^2 - 4x - 60}{(5x - 2)^2}$

**6–12)**  $f'(x) = \frac{x^2 + 2x - 1}{(x + 1)^2}$

$$\mathbf{6-13)} \quad f'(x) = 12x^{11}e^x + x^12e^x$$

$$\mathbf{6-14)} \quad f'(x) = 6x \ln x + 3x$$

$$\mathbf{6-15)} \quad f'(x) = \frac{5 \ln x - 5}{\ln^2 x}$$

$$\mathbf{6-16)} \quad f'(x) = \frac{e^x(1+x^2-2x)}{(1+x^2)^2}$$

$$\mathbf{6-17)} \quad f'(x) = -\frac{2+3e^x}{(1+2x+3e^x)^2}$$

$$\mathbf{6-18)} \quad f'(x) = -\frac{e^x + \frac{2}{x}}{(e^x + 2 \ln x)^2}$$

$$\mathbf{6-19)} \quad f'(x) = x^3 e^x (4 \ln x + x \ln x + 1)$$

$$\mathbf{6-20)} \quad f'(x) = (1 + e^x)(x^2 + \ln x) + (x + e^x) \left( 2x + \frac{1}{x} \right)$$

$$\mathbf{6-21)} \quad f'(x) = \frac{(8x-5)(2e^x-3x)-(2e^x-3)(4x^2-5x)}{(2e^x-3x)^2}$$

$$\mathbf{6-22)} \quad f'(x) = -\frac{1}{x \ln^2(4x)}$$

$$\mathbf{6-23)} \quad f'(x) = 3e^{3x}$$

$$\mathbf{6-24)} \quad f'(x) = -\frac{13}{(5 \ln x + 1)^2}$$

$$\mathbf{6-25)} \quad f'(x) = 8(1 + x^4)x^3$$

$$\mathbf{6-26)} \quad f'(x) = 3x^2e^{x^3}$$

$$\mathbf{6-27)} \quad f'(x) = \frac{2x}{1 + x^2}$$

$$\mathbf{6-28)} \quad f'(x) = 7(5 + x + 2x^3)^6(1 + 6x^2)$$

$$\mathbf{6-29)} \quad f'(x) = \frac{2}{(3x^2 + 2)^{3/2}}$$

$$\mathbf{6-30)} \quad f'(x) = \frac{12 - 6x}{(x^2 - 4x)^4}$$

$$\mathbf{6-31)} \quad f'(x) = 5 \left( \frac{2x}{x - 1} \right)^4 \frac{-2}{(x - 1)^2} = -\frac{160x^4}{(x - 1)^6}$$

$$\mathbf{6-32)} \quad f'(x) = 15(e^{3x} + 1)^4 e^{3x}$$

$$\mathbf{6-33)} \quad f'(x) = \frac{1}{2x\sqrt{1 + \ln x}}$$

$$\mathbf{6-34)} \quad f'(x) = \frac{2x + 6e^{3x}}{2\sqrt{x^2 + 2e^{3x}}}$$

$$\mathbf{6-35)} \quad f'(x) = 4^{x^2+5x}(2x + 5)$$

$$\mathbf{6-36)} \quad f'(x) = (e^x + xe^x)\log_3(x + x^4) + xe^x \frac{1 + 4x^3}{(x + x^4)\ln 3}$$

$$6-37) \quad f''(x) = (4 \ln^2 5) 5^{2x}$$

$$6-38) \quad f''(x) = -\frac{1}{x^2}$$

$$6-39) \quad f''(x) = \frac{1}{4(2+x)^{3/2}}$$

$$6-40) \quad f''(x) = (42x^5 - 14x^6 + x^7) e^{-x}$$

$$6-41) \quad f'(x) = (1+2x)^7 (x^3+1)^4 \left[ \frac{14}{1+2x} + \frac{12x^2}{x^3+1} \right]$$

$$6-42) \quad f'(x) = \frac{(3x^4+x^2)^6}{(1+x+x^2)^8} \left[ \frac{6(12x^3+2x)}{3x^4+x^2} - \frac{8(1+2x)}{1+x+x^2} \right]$$

$$6-43) \quad f'(x) = (\ln x)^x \left[ \ln(\ln x) + \frac{1}{\ln x} \right]$$

$$6-44) \quad y = -4$$

$$6-45) \quad y = 2x$$

$$6-46) \quad y = -4x + 12$$

$$6-47) \quad y = 1$$

$$\mathbf{6-48)} \quad f'(2) = -1$$

$$\mathbf{6-49)} \quad f'(32) = \frac{3}{20}$$

$$\mathbf{6-50)} \quad f'(0) = -\frac{5}{16}$$

$$\mathbf{6-51)} \quad f'(1) = 1$$

$$\mathbf{6-52)} \quad f'(0) = 3$$

$$\mathbf{6-53)} \quad f'(0) = \frac{1}{6}$$

$$\mathbf{6-54)} \quad f'(5) = \frac{1}{12}$$

$$\mathbf{6-55)} \quad f'\left(\frac{1}{2}\right) = -140$$

$$\mathbf{6-56)} \quad f'(0) = 27$$

$$\mathbf{6-57)} \quad f'(0) = -\frac{1}{5}$$

$$\mathbf{6-58)} \quad f'(1) = \frac{1}{2} \ln 3 + \frac{1}{3}$$

$$\mathbf{6-59)} \quad f'\left(\frac{1}{4}\right) = -\frac{e^4}{2}$$

$$\mathbf{6-60)} \quad f'(-1) = \frac{7}{10}$$