

# Math 111 – 7

## Implicit Differentiation, L'Hôpital's Rule

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### 7.1 Implicit Differentiation

An equation involving  $x$  and  $y$  may define  $y$  as a function of  $x$ . This is called an implicit function. For example, the following equations define  $y$  implicitly.

- $x^2 + y^2 = 1,$
- $ye^y + 2x - \ln y = 0,$
- $3xy + x^2y^3 + x = 5,$
- $e^x + e^y = \sqrt{x+2y},$

The following equations define  $y$  explicitly.

- $y = x^3 - 5x^2,$
- $y = \ln(x^2 - e^x),$
- $y = x^3 + \sqrt{x} + xe^x,$
- $y = \frac{1}{1 + e^{x^2-x}},$

The derivative of  $y$  can be found without solving for  $y$ . This is called implicit differentiation. The main idea is:

- Differentiate with respect to  $x$ .
- Solve for  $y'$ .

**Example 7–1:** Find the slope of the tangent line to the curve

$$x^2 + y^2 = 4 \quad \text{at the point} \quad (1, \sqrt{3})$$

**Solution:** Let's differentiate both sides with respect to  $x$ :

$$x^2 + y^2 = 4$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(4)$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$2yy' = -2x$$

$$y' = -\frac{x}{y}$$

Therefore at the point  $(1, \sqrt{3})$ :

$$y' = -\frac{1}{\sqrt{3}}$$

An alternative method is to express  $y$  in terms of  $x$  explicitly as

$$y = \sqrt{4 - x^2}$$

and then differentiate, but usually this is not possible.

**Remark:** Note that we are using chain rule here. For example, derivative of  $y^n$  can be found as:

$$\begin{aligned}\frac{d(y^n)}{dx} &= \frac{d(y^n)}{dy} \frac{dy}{dx} \\ &= ny^{n-1} y'\end{aligned}$$

**Example 7–2:** Find the slope of the tangent line to the curve

$$x^8 + 4x^2y^2 + y^8 = 6 \quad \text{at the point} \quad (1, 1)$$

**Solution:** Using implicit differentiation we obtain:

$$8x^7 + 8xy^2 + 8x^2yy' + 8y^7y' = 0$$

$$x^7 + xy^2 + (x^2y + y^7)y' = 0$$

$$\Rightarrow y' = \frac{-x^7 - xy^2}{x^2y + y^7}$$

$$\text{At } (1, 1) \text{ the slope is: } y' = \frac{-2}{2} = -1.$$

**Example 7–3:** Find  $y'$  at  $(0, 0)$  where

$$(1 + x + 2y)e^y + 3xe^x = 1 + x^2 + y^2$$

**Solution:** Using implicit differentiation we obtain:

$$(1 + 2y')e^y + (1 + x + 2y)e^y y' + 3e^x + 3xe^x = 2x + 2yy'$$

$$(2e^y + (1 + x + 2y)e^y - 2y)y' = 2x - e^y - 3e^x - 3xe^x$$

$$y' = \frac{2x - e^y - 3(1 + x)e^x}{(3 + x + 2y)e^y - 2y}$$

$$y'(0, 0) = -\frac{4}{3}.$$

## 7.2 L'Hôpital's Rule

Some limits like  $\frac{0}{0}$ ,  $\frac{\infty}{\infty}$ ,  $\infty - \infty$ , ... etc. are called indeterminate forms. These limits may turn out to be definite numbers, or infinity, or may not exist.

**Example 7–4:** What is the limit  $\frac{0}{0}$ ?

**Solution:** This is an indeterminate form. If we divide two functions where both of them approach zero, the result could be anything. For example:

$$\lim_{x \rightarrow 0} \frac{x^5}{x^2} = 0, \quad \lim_{x \rightarrow 0} \frac{x^5}{x^7} = \infty, \quad \lim_{x \rightarrow 0} \frac{3x^5}{4x^5} = \frac{3}{4}.$$

**L'Hôpital's Rule:** Consider the limit

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

Assume that we have an indeterminate form of the type:

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

Suppose  $g'(x) \neq 0$  on an open interval containing  $a$  (except possibly at  $x = a$ ).

Then:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if this limit exists, or is  $\pm\infty$ .

**Example 7–5:** Evaluate the limit  $\lim_{x \rightarrow \infty} \frac{e^x}{x^3}$ .

**Solution:** Indeterminacy of the form  $\frac{\infty}{\infty} \Rightarrow$  use L'Hôpital.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{e^x}{x^3} &= \lim_{x \rightarrow \infty} \frac{e^x}{3x^2} \\&= \lim_{x \rightarrow \infty} \frac{e^x}{6x} \\&= \lim_{x \rightarrow \infty} \frac{e^x}{6} \\&= \infty\end{aligned}$$

The result would be the same if it were  $x^{30}$  rather than  $x^3$ . Exponential function increases faster than all polynomials.

**Example 7–6:** Evaluate the limit  $\lim_{x \rightarrow 1} \frac{x^{10} - 1}{x^7 - 1}$ .

**Solution:** It is possible to solve this question using the algebraic identities:

$$x^{10} - 1 = (x - 1)(x^9 + x^8 + \cdots + x + 1)$$

$$x^7 - 1 = (x - 1)(x^6 + x^5 + \cdots + x + 1)$$

but this is too complicated. Limit is in the form  $\frac{0}{0}$  and using L'Hôpital gives the same result easily.

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^{10} - 1}{x^7 - 1} &= \lim_{x \rightarrow 1} \frac{10x^9}{7x^6} \\&= \frac{10}{7}\end{aligned}$$

**Example 7–7:** Evaluate the following limit: (if it exists.)

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3 + x^4}$$

**Solution:** This limit is in the form  $\frac{0}{0}$ , so using L'Hôpital's rule we obtain:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3 + x^4} &= \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{3x^2 + 4x^3} \\ &= \lim_{x \rightarrow 0} \frac{e^x - 1}{6x + 12x^2} \\ &= \lim_{x \rightarrow 0} \frac{e^x}{6 + 24x}\end{aligned}$$

At this point, the limit is NOT in the form  $\frac{0}{0}$ , so we can NOT use L'Hôpital. Just insert  $x = 0$  to obtain:

$$= \frac{1}{6}$$

**Example 7–8:** Evaluate the limit  $\lim_{x \rightarrow \infty} \frac{\ln x + x^2}{xe^x}$ .

**Solution:** Indeterminacy of the form  $\frac{\infty}{\infty} \Rightarrow$  Use L'Hôpital:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\ln x + x^2}{xe^x} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + 2x}{e^x + xe^x} \\ &= \lim_{x \rightarrow \infty} \frac{-\frac{1}{x^2} + 2}{e^x + e^x + xe^x} \\ &= 0\end{aligned}$$

## EXERCISES

Find  $y'$  using implicit differentiation:

$$7-1) \quad x^2y^3 + 3xy^2 + y = 5$$

$$7-2) \quad xye^x + (x + 2y)^2 = x$$

$$7-3) \quad (x^2 + y)^2 = y^3$$

$$7-4) \quad x = y + y^{2/3}$$

$$7-5) \quad (1 + e^{-x})^2 = \ln(x + y)$$

$$7-6) \quad \ln y = y^3 + \ln x$$

$$7-7) \quad e^{xy} = x + 2y$$

$$7-8) \quad x^2 + \ln y = 3xy$$

$$7-9) \quad y^2 \ln y = x^3 e^x$$

$$7-10) \quad \sqrt{5x + y^3} + xy = 12$$

$$7-11) \quad \frac{2}{x} + \frac{7}{y} = 9$$

$$7-12) \quad x^{1/3} + y^{1/5} = y$$

Find  $y'$  at the indicated point:

**7-13)**  $x^2y^2 + 2xy^3 + y - 10x + 11 = 0$  at  $(2, 1)$

**7-14)**  $xy^4 + 3y^5 + x - 3y^3 = 0$  at  $(0, 1)$

**7-15)**  $2x - 4y^4 + x^2y^6 + 11y^3 = 0$  at  $(3, -1)$

**7-16)**  $\sqrt{11 + y^2} - 12xy + 2y^2 + 4x = 0$  at  $(1, 5)$

**7-17)**  $xe^x - ye^y + xy - 1 = 0$  at  $(1, 1)$

**7-18)**  $\ln(xy) + xy^2 - \ln 3x - 6y = 0$  at  $(2, 3)$

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Evaluate the following limits:

**7-19)**  $\lim_{x \rightarrow 0} \frac{e^{5x} - e^{4x} - x}{x^2}$

**7-20)**  $\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{\ln(x + 1)}$

**7-21)**  $\lim_{x \rightarrow 0} \frac{e^x - 1}{e^{2x} + 3x - 1}$

**7-22)**  $\lim_{x \rightarrow \infty} \frac{3x^2 + 4 \ln x}{6x^2 + 7 \ln x}$

**7-23)**  $\lim_{x \rightarrow \infty} \frac{2e^x + 5x}{7e^x + 8x + 12}$

**7-24)**  $\lim_{x \rightarrow \infty} \frac{\ln(x + x^4)}{x}$

Evaluate the following limits:

$$7-25) \lim_{x \rightarrow 64} \frac{x^{1/3} - 4}{x^{1/2} - 8}$$

$$7-26) \lim_{x \rightarrow 0} \frac{\sqrt{9 + 2x} - 3}{\sqrt{16 + x} - 4}$$

$$7-27) \lim_{x \rightarrow 1/2} \frac{\ln(2x)}{2x^2 + x - 1}$$

$$7-28) \lim_{x \rightarrow e} \frac{\ln x - 1}{x - e}$$

$$7-29) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$$

$$7-30) \lim_{x \rightarrow 3} \frac{x^3 - 4x - 15}{x^2 + x - 12}$$

$$7-31) \lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}}$$

$$7-32) \lim_{x \rightarrow 1} \frac{x^6 - 1}{x^4 - 1}$$

$$7-33) \lim_{x \rightarrow 0} \frac{\sqrt{a+bx} - \sqrt{a+cx}}{x}$$

$$7-34) \lim_{x \rightarrow 0} \frac{4^x - 1}{2^x - 1}$$

$$7-35) \lim_{x \rightarrow 2} \frac{\ln \frac{x}{2}}{x(x-2)}$$

$$7-36) \lim_{x \rightarrow 0^+} x \ln x$$

# ANSWERS

$$7-1) \quad y' = -\frac{2xy^3 + 3y^2}{3x^2y^2 + 6xy + 1}$$

$$7-2) \quad y' = -\frac{ye^x + xye^x + 2x + 4y - 1}{xe^x + 4x + 8y}$$

$$7-3) \quad y' = \frac{4x^3 + 4xy}{3y^2 - 2x^2 - 2y}$$

$$7-4) \quad y' = \frac{1}{1 + \frac{2}{3}y^{-1/3}}$$

$$7-5) \quad y' = -2(x + y)e^{-x}(1 + e^{-x}) - 1$$

$$7-6) \quad y' = \frac{y}{x(1 - 3y^3)}$$

$$7-7) \quad y' = \frac{1 - ye^{xy}}{xe^{xy} - 2}$$

$$7-8) \quad y' = \frac{3y^2 - 2xy}{1 - 3xy}$$

$$7-9) \quad y' = \frac{3x^2e^x + x^3e^x}{2y \ln y + y}$$

$$7-10) \quad y' = -\frac{2y\sqrt{5x + y^3} + 5}{2x\sqrt{5x + y^3} + 3y^2}$$

$$7-11) \quad y' = -\frac{2y^2}{7x^2}$$

$$7-12) \quad y' = \frac{5y^{4/5}}{3x^{2/3}(5y^{4/5} - 1)}$$

$$7-13) \quad m = \frac{4}{21}$$

$$7-14) \quad m = -\frac{1}{3}$$

$$7-15) \quad m = \frac{8}{5}$$

$$7-16) \quad m = \frac{336}{53}$$

$$7-17) \quad m = \frac{2e+1}{2e-1}$$

$$7-18) \quad m = -\frac{27}{19}$$

$$7-19) \quad \frac{9}{2}$$

$$7-20) \quad 3$$

$$7-21) \quad \frac{1}{5}$$

$$7-22) \quad \frac{1}{2}$$

$$7-23) \quad \frac{2}{7}$$

$$7-24) \quad 0$$

$$7-25) \quad \frac{1}{3}$$

$$7-26) \quad \frac{8}{3}$$

$$7-27) \quad \frac{2}{3}$$

$$7-28) \quad \frac{1}{e}$$

$$7-29) \quad \frac{1}{2}$$

$$7-30) \quad \frac{23}{7}$$

$$7-31) \quad 0$$

$$7-32) \quad \frac{3}{2}$$

$$7-33) \quad \frac{b-c}{2\sqrt{a}}$$

$$7-34) \quad 2$$

$$7-35) \quad \frac{1}{4}$$

$$7-36) \quad 0$$