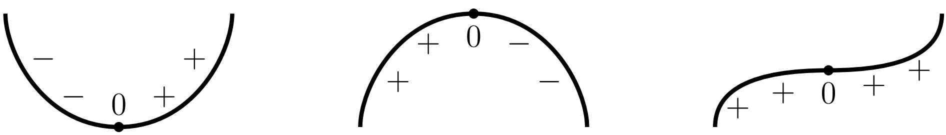
Math 111 – 8
Curve Sketching

8.1 First Derivative Test

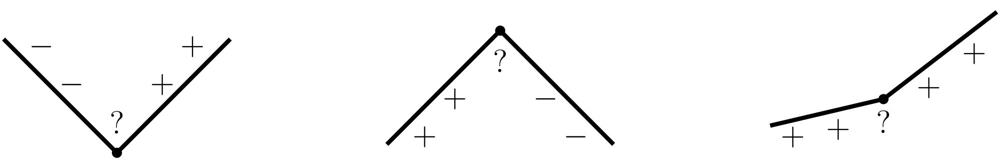
At a critical point, the derivative is zero or undefined. Let f be a continuous function and let $x = c$ be a critical point of it. Suppose f' exists in some interval containing c except possibly at c.

f has a local extremum at c if and only if f' changes sign at c.

- Sign change: $-$ to $+$ $\quad\Rightarrow\quad$ $f(c)$ is a local minimum.
- Sign change: $+$ to $-$ $\quad\Rightarrow\quad$ $f(c)$ is a local maximum.



Local Min. Local Max. Neither



At sharp points, derivative is not defined. The curve has two different tangents from left and right. But the function is defined.

Example 8–1: Find the intervals where $f(x) = 2x^3 - 9x^2 + 5$ is increasing and decreasing. Then find all the local extrema of this function using first derivative test.

Solution: $f'(x) = 6x^2 - 18x = 0$

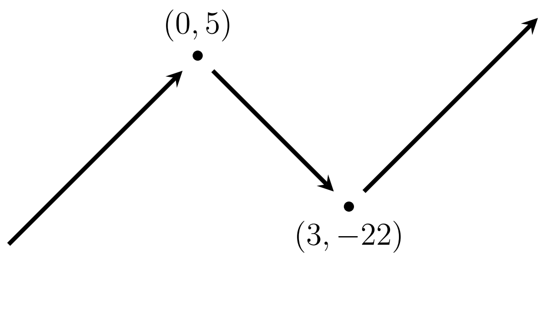
$$6x(x-3) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = 3.$$

There are two critical points, 0 and 3. Note that $f(0) = 5$ and $f(3) = -22$.

x changes sign at 0 and $(x-3)$ changes sign at 3. We can find the sign of $x(x-3)$ by multiplying these signs.

x		0		3	
$(x-3)$	$-$		$-$	0	$+$
$f' = x(x-3)$	$+$	0	$-$	0	$+$
f	increasing		decreasing		increasing

Based on this table, we can see that the graph is roughly like this:



Therefore $(0, 5)$ is local maximum and $(3, -22)$ is local minimum.

Example 8–2: Find the intervals where $f(x) = x^{\frac{4}{5}}$ is increasing and decreasing. Then find all the local extrema of this function using first derivative test.

Solution: Derivative gives:

$$f'(x) = \frac{4}{5}\,x^{-\frac{1}{5}}$$

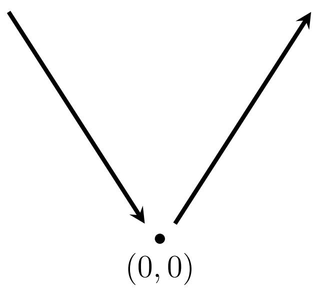
In other words: $f' = \dfrac{4}{5\,\sqrt[5]{x}}$

f' is never zero. But it is undefined at $x = 0$. This is the only critical point.

Note that $\sqrt[5]{x}$ is negative when x is negative.

x		0	
f'	$-$	$\Vert$	$+$
f	$\searrow$		$\nearrow$

Based on this table, we can see that the graph is roughly like this:



By first derivative test, $(0, 0)$ is local minimum.

Example 8–3: Using first derivative test, find local extrema of

$$f(x) = x^4 - 4x^3 - 2x^2 + 12x$$

Solution: $f'(x) = 4x^3 - 12x^2 - 4x + 12 = 0$. Using trial and error we find that $x = 1$ is a solution, therefore:

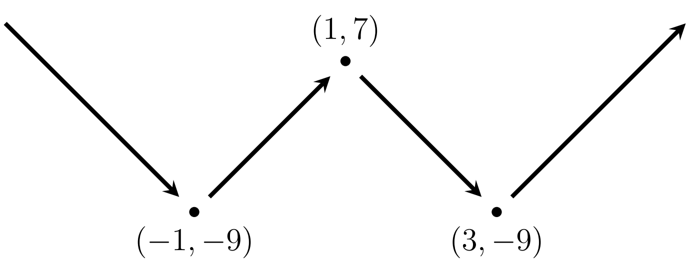
$$(x-1)(x^2-2x-3) = 0 \quad \Rightarrow \quad (x-1)(x+1)(x-3) = 0.$$

Critical points are $x = -1, \quad x = 1$ and $x = 3$. Note that $f(-1) = -9, \quad f(1) = 7$ and $f(3) = -9$.

To find the sign of f' over these intervals, we need a table:

x		-1		1		3	
$(x+1)$	$-$	0	$+$		$+$		$+$
$(x-1)$	$-$		$-$	0	$+$		$+$
$(x-3)$	$-$		$-$		$-$	0	$+$
f'	$-$	0	$+$	0	$-$	0	$+$
f	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

Based on the table, we obtain:



Therefore $(-1, -9)$ is local minimum, $(1, 7)$ is local maximum, and $(3, -9)$ is local minimum.

8.2 Concavity

The graph of a differentiable function is concave up if f' increasing, it is concave down if f' decreasing.

Test for Concavity:

- If $f''(x) > 0$, then f is concave up at x.
- If $f''(x) < 0$, then f is concave down at x.

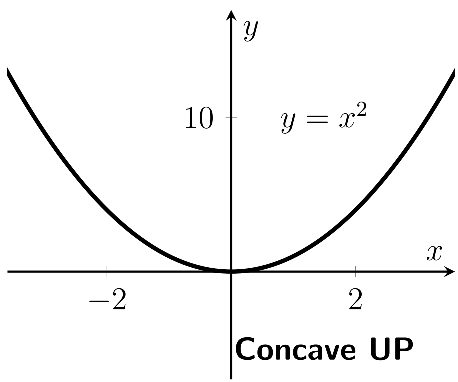
Inflection Point: An inflection point is a point where the concavity changes. In other words, if:

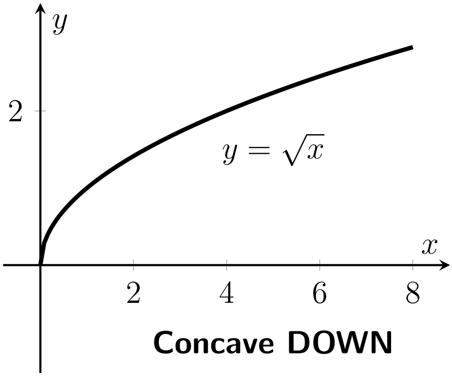
- f is continuous at $x = a$,
- $f'' > 0$ on the left of a and $f'' < 0$ on the right, or vice versa.

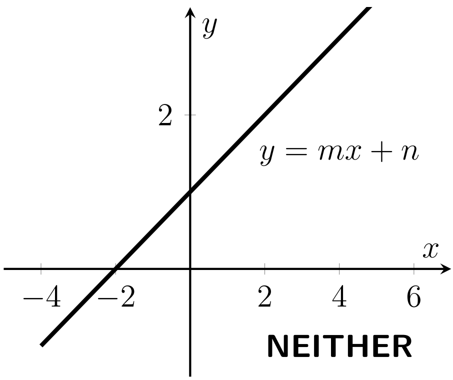
then $x = a$ is an inflection point.

This means either $f''(a) = 0$ or $f''(a)$ does not exist.

Examples:




y
2
$y = \sqrt{x}$
x
2
4
6
8
Concave DOWN


y
2
$y = mx + n$
x
-4
-2
2
4
6
NEITHER

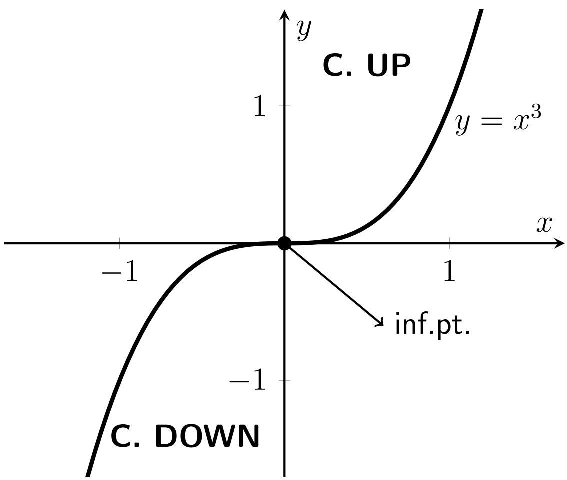
Example 8–4: Determine the concavity of $f(x) = x^3$. Find inflection points. (If there is any.)

Solution:

$$\begin{aligned} f &= x^3 \\ f' &= 3x^2 \\ f'' &= 6x \end{aligned}$$

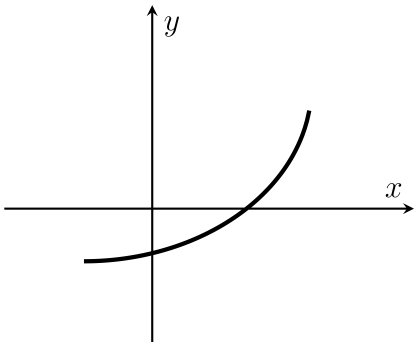
- For $x > 0$, $f'' > 0 \quad \Rightarrow \quad f$ is concave up.
- For $x < 0$, $f'' < 0 \quad \Rightarrow \quad f$ is concave down.
- $x = 0$ is the inflection point.

x		0	
f''	$-$	0	$+$
f is:	concave down		concave up



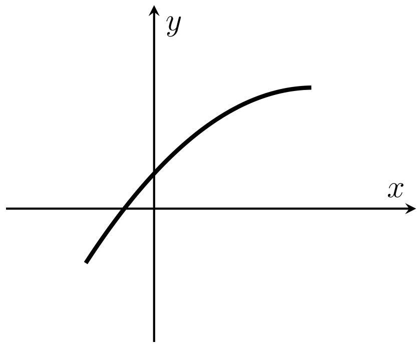
Shape of a graph based on first and second derivatives:

$f' > 0, \quad f'' > 0$



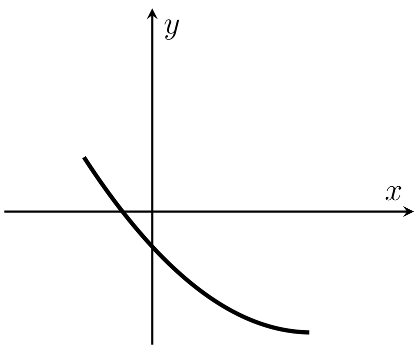
Increasing, Concave up.

$f' > 0, \quad f'' < 0$



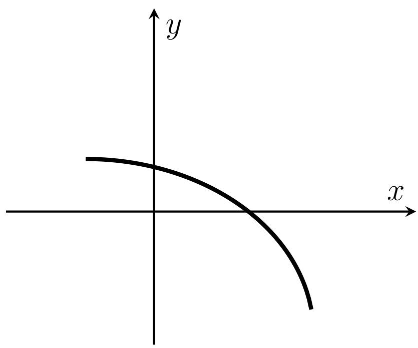
Increasing, Concave down.

$f' < 0, \quad f'' > 0$



Decreasing, Concave up.

$f' < 0, \quad f'' < 0$



Decreasing, Concave down.

8.3 Curve Sketching

- Identify domain of f, symmetries, x and y intercepts. (if any)
- Find first and second derivatives of f.
- Find critical points, inflections points.
- Make a table and include all this information.
- Sketch the curve using the table.

Example 8–5: Sketch the graph of $f(x) = x^3 + 3x^2 - 24x$.

Solution: $\lim_{x\to\infty} f = +\infty, \quad \lim_{x\to-\infty} f = -\infty$

$$\begin{aligned} f' &= 3x^2 + 6x - 24 \\ &= 3(x+4)(x-2) \end{aligned}$$

$f' = 0 \quad \Rightarrow \quad x = -4, \quad$ and $\quad x = 2.$

These are the critical points.

$f'' = 6x + 6 = 0 \quad \Rightarrow \quad x = -1.$

This is the inflection point.

Some specific points on the graph are:

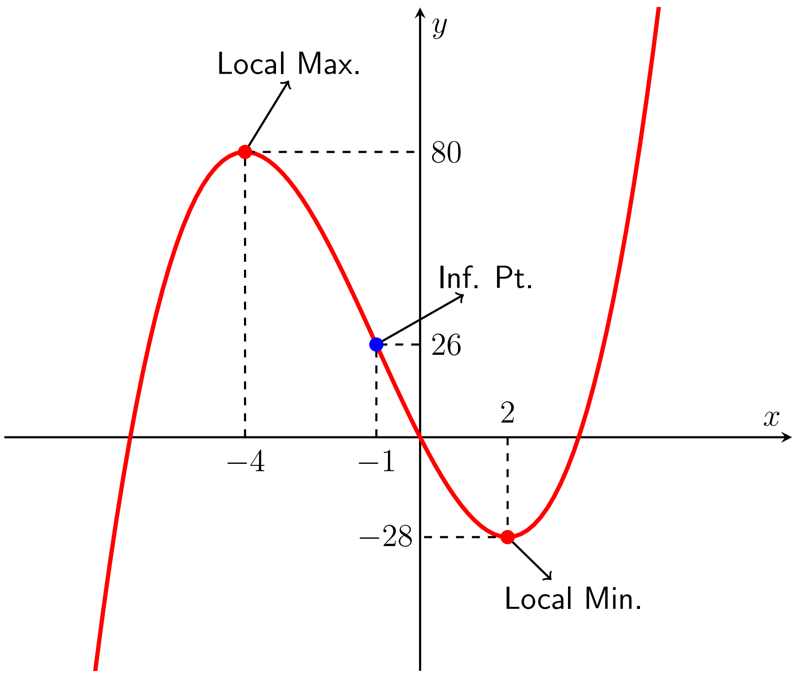
$f(-4) = 80, \quad f(-1) = 26.$

$f(0) = 0, \quad f(2) = -28.$

The equation $f(x) = 0$ gives $x = 0$ or $x^2 + 3x - 24 = 0$ in other words $x = \dfrac{-3 \pm \sqrt{105}}{2}$. Using a calculator we find $x_1 = -6.6,\ x_2 = 3.6$ but it is possible to sketch the graph without these points. Putting all this information on a table, we obtain:

x		-4		-1		2	
f'	$+$	0	$-$		$-$	0	$+$
f''	$-$		$-$	0	$+$		$+$
f	$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

Based on this table, we can sketch the graph as:



EXERCISES

Determine the intervals where the following functions are increasing and decreasing:

8–1) $f(x) = x^3 - 12x - 5$

8–2) $f(x) = 16 - 4x^2$

8–3) $f(x) = \dfrac{1}{(x-4)^2}$

8–4) $f(x) = \dfrac{x^2 - 3}{x - 2}$

8–5) $f(x) = 4x^5 + 5x^4 - 40x^3$

8–6) $f(x) = x^4 e^{-x}$

8–7) $f(x) = \dfrac{\ln x}{x}$

8–8) $f(x) = 5x^6 + 6x^5 - 45x^4$

8–9) $f(x) = x^4 - 2x^2 + 1$

8–10) $f(x) = \dfrac{x}{x+1}$

Identify local maxima, minima and inflection points, then sketch the graphs of the following functions:

8–11) $f(x) = x^3 - 3x^2 - 9x + 11$

8–12) $f(x) = -2x^3 + 21x^2 - 60x$

8–13) $f(x) = 3x^4 + 4x^3 - 36x^2$

8–14) $f(x) = (x-1)^2(x+2)^3$

8–15) $f(x) = x^6 - 6x^5$

8–16) $f(x) = x^3 e^{-x}$

8–17) $f(x) = e^{-x^2}$

8–18) $f(x) = \dfrac{x}{x^2+1}$

8–19) $f(x) = x \ln |x|$

8–20) $f(x) = -x^4 + 32x^2$

ANSWERS

8–1) Increasing on $(-\infty, -2)$, decreasing on $(-2, 2)$, increasing on $(2, \infty)$.

8–2) Increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$.

8–3) Increasing on $(-\infty, 4)$, decreasing on $(4, \infty)$.

8–4) Increasing on $(-\infty, 1)$, decreasing on $(1, 2) \cup (2, 3)$, increasing on $(3, \infty)$.

8–5) Increasing on $(-\infty, -3)$, decreasing on $(-3, 0) \cup (0, 2)$, increasing on $(2, \infty)$.

8–6) Decreasing on $(-\infty, 0)$, increasing on $(0, 4)$, decreasing on $(4, \infty)$.

8–7) Increasing on $(0, e)$, decreasing on (e, ∞).

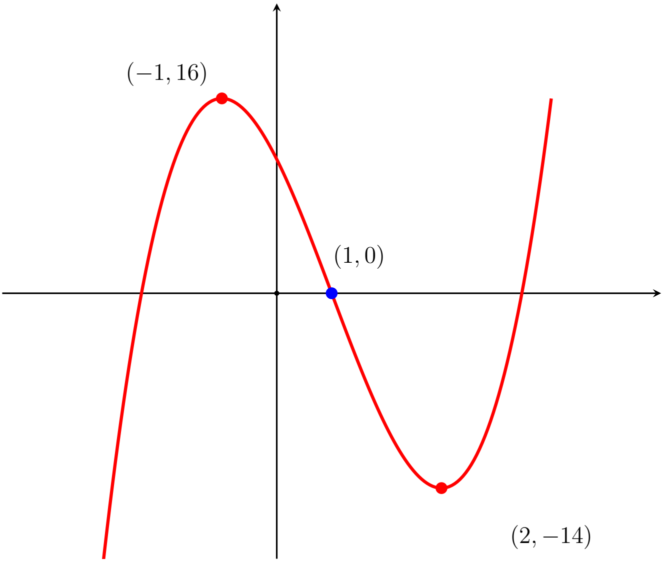
8–8) Decreasing on $(-\infty, -3)$, increasing on $(-3, 0)$, decreasing on $(0, 2)$, increasing on $(2, \infty)$.

8–9) Decreasing on $(-\infty, -1)$, increasing on $(-1, 0)$, decreasing on $(0, 1)$, increasing on $(1, \infty)$.

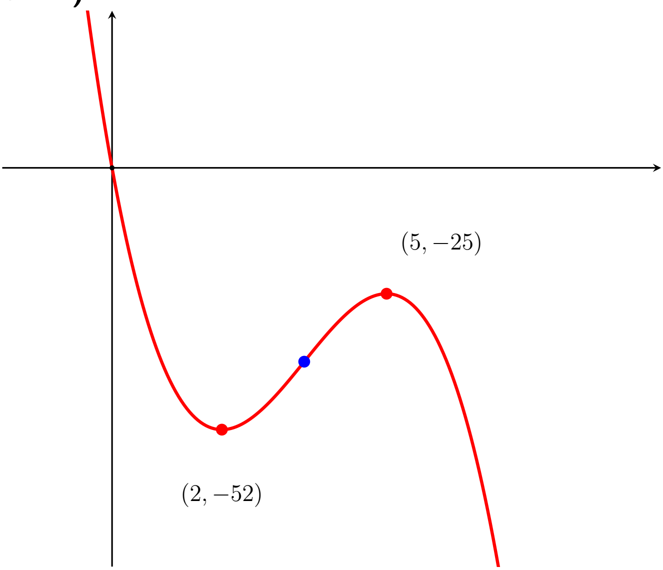
8–10) Increasing on $(-\infty, -1) \cup (-1, \infty)$.

(Blue dots denote inflection points, red dots local extrema.)

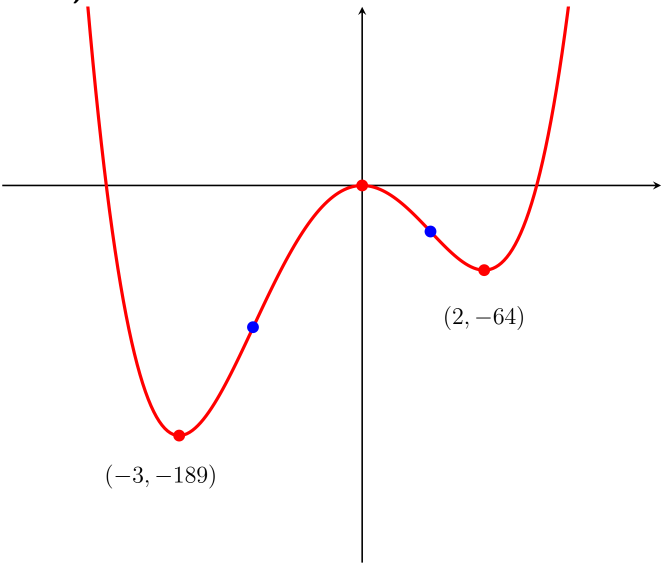
8–11)



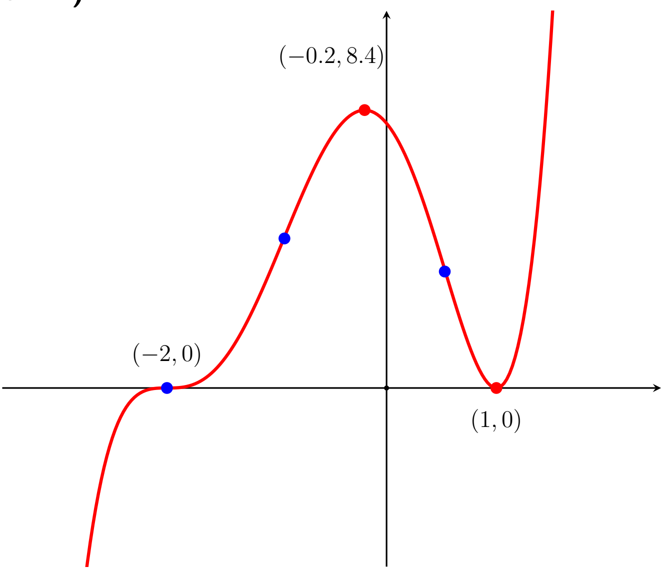
8–12)



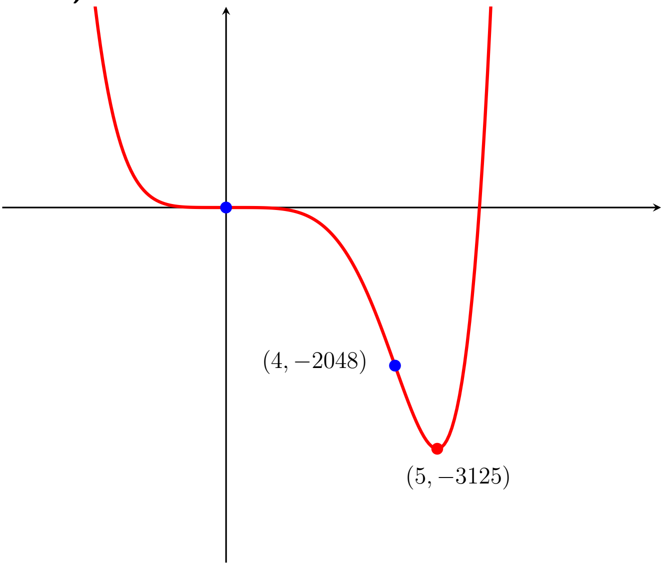
8–13)



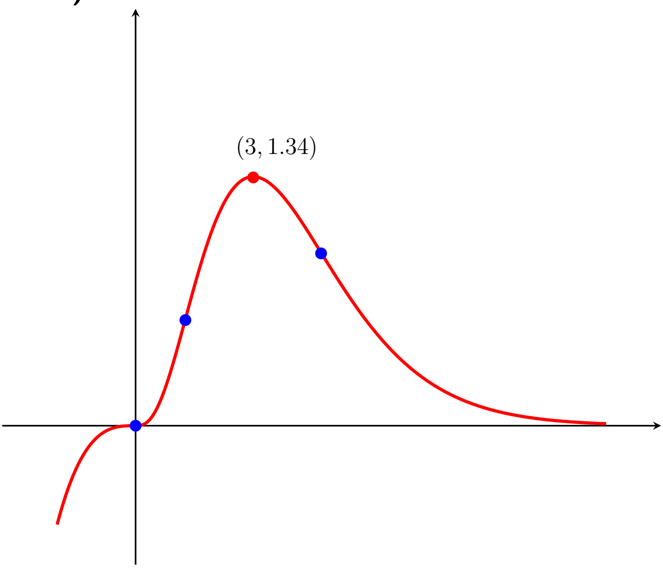
8–14)



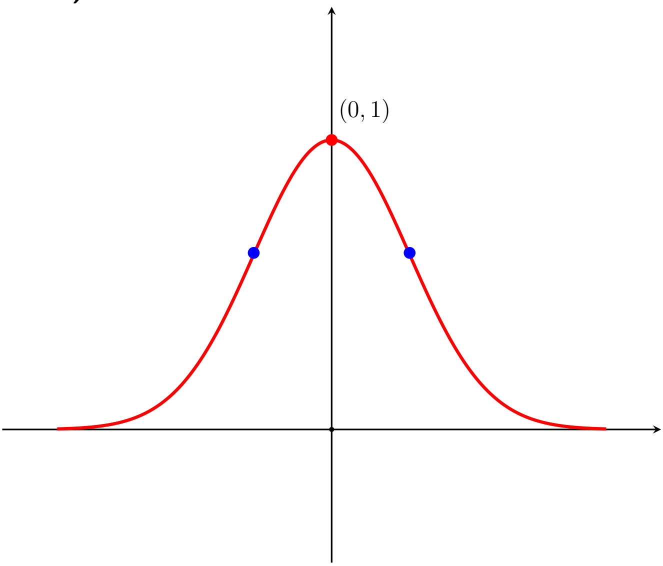
8–15)



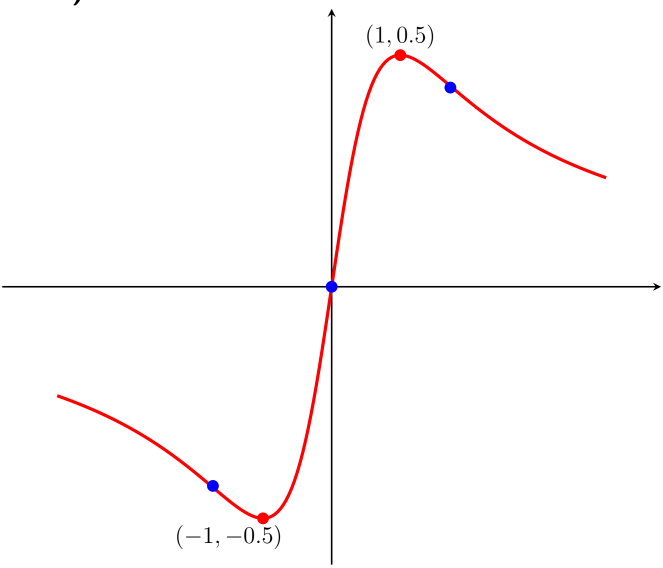
8–16)



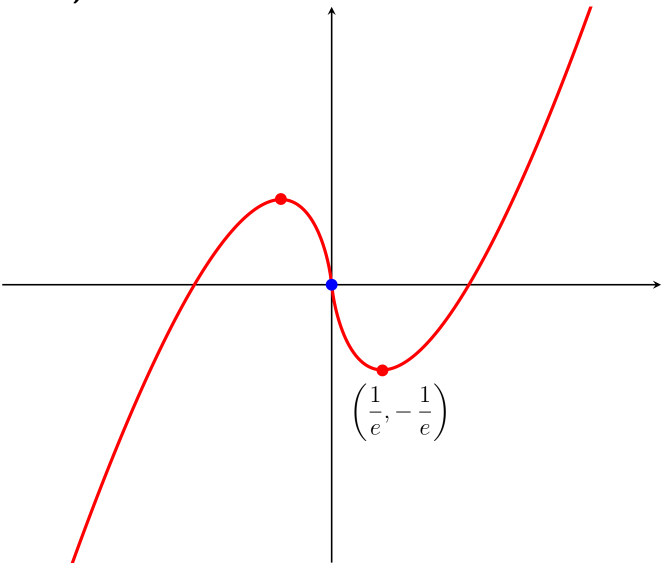
8–17)



8–18)



8–19)



8–20)