Math 111 – 9

Finding Maximum and Minimum Values

9.1 Absolute Extrema

Extremum is either minimum or maximum. Extrema is the plural form.

Absolute Extrema: If

$$f(c) \leqslant f(x)$$

for all x on a set S of real numbers, $f(c)$ is the absolute minimum value of f on S.

Similarly if

$$f(c) \geqslant f(x)$$

for all x on S, $f(c)$ is the absolute maximum value of f on S.

Local Extrema: $f(c)$ is local minimum if

$$f(c) \leqslant f(x)$$

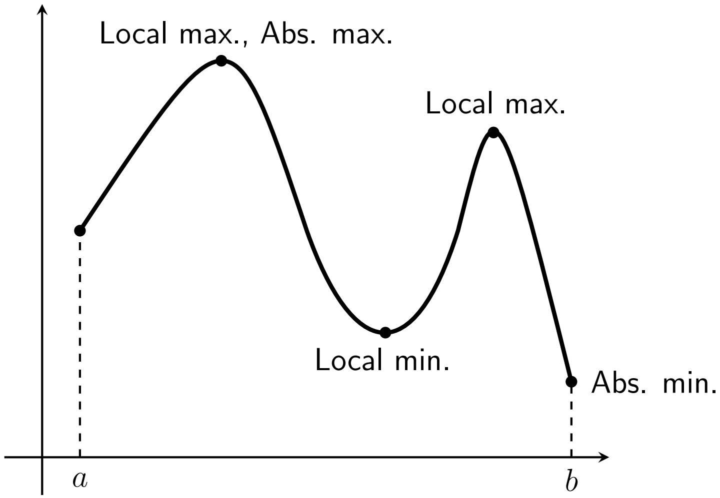
for all x in some open interval containing c.

Similarly, $f(c)$ is local maximum if

$$f(c) \geqslant f(x)$$

for all x in some open interval containing c.

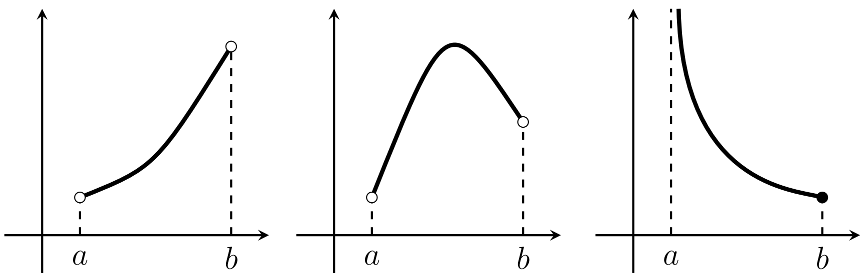
Local extrema are points that are higher (or lower) than the points around them.



As you can see in the figure, a point can be both local and absolute extremum. Also, it may be an absolute extremum without being a local one or vice versa.

Question: Does a continuous function always have an absolute maximum and an absolute minimum value?

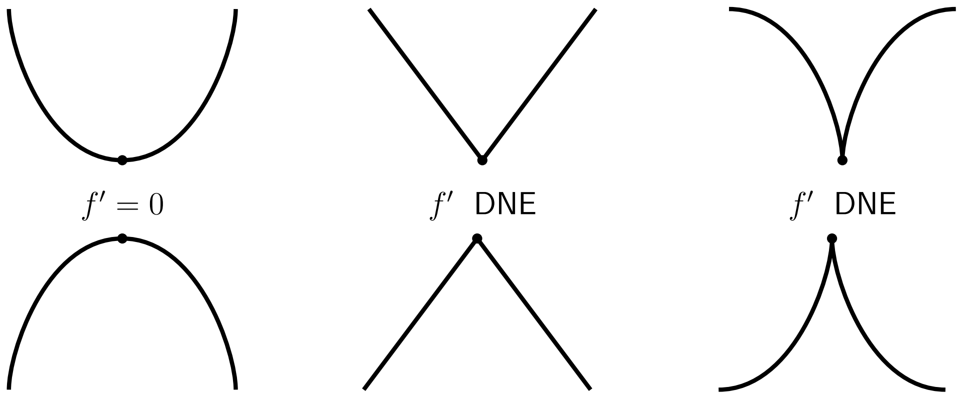
This depends on the interval. It may or may not have such values on an open interval.



No max. or min. Max. but no min. Min. but no max.

Theorem: If the function f is continuous on the closed interval $[a,\, b]$, then f has a maximum and a minimum value on $[a,\, b]$.

Critical Point: A number c is called a critical point of the function f if $f'(c) = 0$ or $f'(c)$ does not exist.



The main ideas about extremum points can be summarized as:

1. f can have local extremum only at a critical point.

2. f can have absolute extremum only at a critical point or an endpoint.

How to find absolute extrema:

- Find the points where $f' = 0$.

- Find the points where f' does not exist.

- Consider such points only if they are **inside** the given interval.

- Consider endpoints.

- Check all candidates. Both absolute minimum and maximum are among them.

Example 9–1: Find the maximum and minimum values of $f(x) = -x^2 + 10x + 2$ on the interval $\left[1,\, 4\right]$.

Solution: Let's find the critical points first:

$f' = -2x + 10 = 0 \quad \Rightarrow \quad x = 5$ is the only critical point. But it is not in our interval $\left[0,\, 4\right]$, so our candidates for extrema are the endpoints:

x	$f(x)$
1	11
4	26

Clearly, absolute minimum is 11 and it occurs at $x = 1$. Absolute maximum is 26 and it occurs at $x = 4$.

Example 9–2: Find the maximum and minimum values of $f(x) = -x^2 + 10x + 2$ on the interval $\left[2,\, 10\right]$.

Solution: Although it is the same function, interval is different.

$f'(x) = -2x + 10 = 0 \quad \Rightarrow \quad x = 5$ is the only critical point. It is inside the interval.

x	$f(x)$
2	18
5	27
10	2

Absolute minimum is 2 and it occurs at $x = 10$. Absolute maximum is 27 and it occurs at $x = 5$.

Example 9–3: Find the maximum and minimum values of

$$f(x) = xe^{-x}$$

on the interval $\big[0,\, 2\big]$.

Solution: Derivative gives:

$$f'(x) = e^{-x} - xe^{-x} = 0$$

$$\Rightarrow \quad e^{-x}\big(1 - x\big) = 0.$$

The only critical point is $x = 1$. Together with the endpoints, we should check all candidates:

x	$f(x)$
0	0
1	$\dfrac{1}{e}$
2	$\dfrac{2}{e^2}$

Clearly, absolute minimum is: $f(0) = 0$.

Even without a calculator, we should be able to see that the absolute maximum is $f(1) = \dfrac{1}{e}$, because:

$$e > 2 \qquad \Rightarrow \qquad \frac{2}{e} < 1$$

$$\Rightarrow \qquad \frac{2}{e} \cdot \frac{1}{e} < \frac{1}{e}$$

$\Bigg($Using a calculator we find $\dfrac{1}{e} = 0.37$ and $\dfrac{2}{e^2} = 0.27.\Bigg)$

Example 9–4: Find the maximum and minimum values of $f(x) = \left|x - 8\right|$ on the interval $\left[6,\, 12\right]$.

Solution: Derivative gives: $f'(x) = 1$ for $x > 8$ and $f'(x) = -1$ for $x < 8$, so derivative is never zero.

The only critical point is $x = 8$. Derivative does not exist at that point.

x	$f(x)$	
6	2	
8	0	Abs. Min.
12	4	Abs. Max.

Example 9–5: Find the maximum and minimum values of $f(x) = \left|16 - x^2\right|$ on the interval $\left[-3,\, 5\right]$.

Solution: First, we have to express this as a piecewise defined function:

$$f(x) = \begin{cases} x^2 - 16 & \text{if} \quad x < -4 \\ 16 - x^2 & \text{if} \quad -4 \leqslant x \leqslant 4 \\ x^2 - 16 & \text{if} \quad x \geqslant 4 \end{cases}$$

Derivative is zero at $x = 0$ and derivative is undefined at $x = \pm 4$. We will not consider $x = -4$ because it is outside the interval.

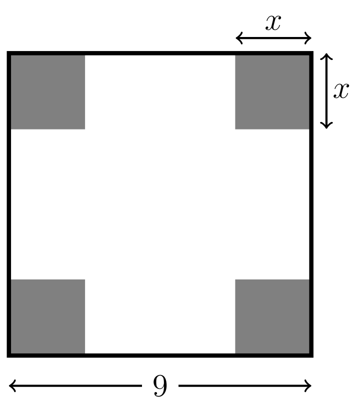
x	$f(x)$	
-3	7	
0	16	Abs. Max.
4	0	Abs. Min.
5	9	

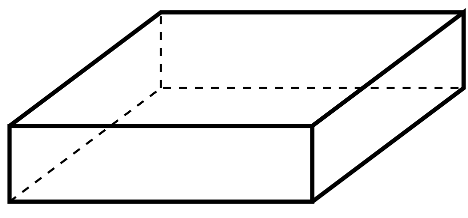
9.2 Applied Optimization

Finding the maximum or minimum of a function has many real-life applications. For these problems:

- Express the quantity to be maximized or minimized as a function of the independent variable. (We will call it x)
- Determine the interval over which x changes.
- Solve the problem in the usual way. (Find the critical points, check the function at critical points and endpoints)

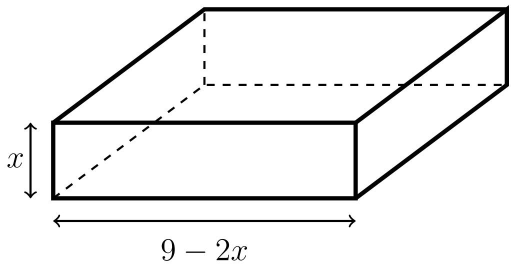
Example 9–6: A piece of cardboard is shaped as a 9×9 square. We will cut four small squares from the corners and make an open top box. What is the maximum possible volume of the box?





Solution: If the squares have edge length x, we can express the volume as:

$$V(x) = x(9-2x)^2 = 81x - 36x^2 + 4x^3$$



Considering the maximum and minimum possible values, we can see that $x \in \left[0, \frac{9}{2}\right]$. Now we can use maximization procedure:

$$V'(x) = 81 - 72x + 12x^2 = 0$$

$$27 - 24x + 4x^2 = 0$$

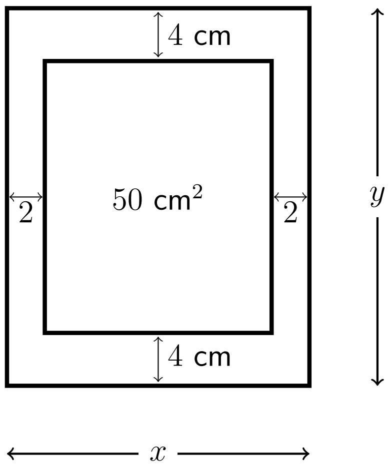
$$(2x-9)(2x-3) = 0$$

$$x = \frac{9}{2} \quad \text{or} \quad x = \frac{3}{2}$$

Checking all critical and endpoints, we find that $x = \dfrac{3}{2}$ gives the maximum volume, which is:

$$V = 54.$$

Example 9–7: You are designing a rectangular poster to contain 50 cm^2 of picture area with a 4 cm margin at the top and bottom and a 2 cm margin at each side. Find the dimensions x and y that will minimize the total area of the poster.



Solution: $(x-4)(y-8)=50 \quad \Rightarrow \quad y=\dfrac{50}{x-4}+8$

$$A = xy = x\left(\frac{50}{x-4}+8\right)$$

$$A' = \frac{50}{x-4}+8-\frac{50x}{(x-4)^2}=0$$

$$\frac{200}{(x-4)^2}=8 \quad \Rightarrow \quad (x-4)^2=25$$

$$\Rightarrow \quad x=9 \quad \text{and} \quad y=18.$$

Example 9–8: You are selling tickets for a concert. If the price of a ticket is $\$15$, you expect to sell 600 tickets. Market research reveals that, sales will increase by 40 for each $\$0.5$ price decrease, and decrease by 40 for each $\$0.5$ price increase. For example, at $\$14.5$ you will sell 640 tickets. At $\$16$ you will sell 520 tickets.

What should the ticket price be for largest possible revenue?

Solution: We need to define our terms first:

- x denotes the sale price of a ticket in $\$$,
- N denotes the number of tickets sold,
- R denotes the revenue.

According to market research, $N = 600 + 40\,\dfrac{15-x}{0.5}$.
In other words:

$$N = 600 + 80\big(15 - x\big) = 1800 - 80x.$$

Note that we sell zero tickets if $x = \dfrac{1800}{80} = 22.5$.
(That's the highest possible price.)

Revenue is:
$$\begin{aligned} R &= Nx \\ &= (1800 - 80x)x \\ &= 1800x - 80x^2 \end{aligned}$$

This is a maximization problem where the interval of the variable is: $x \in \big[0,\, 22.5\big]$.

$$R' = 1800 - 160x = 0 \quad \Rightarrow \quad x = 11.25$$

Checking the critical point $x = 11.25$ and endpoints

0 and 22.5 we see that the maximum revenue occurs at $x = 11.25$.

Example 9–9: A helicopter will cover a distance of 235 km. with constant speed v km/h. The amount of fuel used during flight in terms of liters per hour is

$$75 + \frac{v}{3} + \frac{v^2}{1200}.$$

Find the speed v that minimizes total fuel used during flight.

Solution: The time it takes for flight is: $t = \dfrac{235}{v}$

The total amount of fuel consumed is:

$$\left(75 + \frac{v}{3} + \frac{v^2}{1200}\right) \cdot t = 235 \cdot \left(\frac{75}{v} + \frac{1}{3} + \frac{v}{1200}\right)$$

In other words we have to find v that minimizes $f(v)$ on $v \in \big(0, \infty\big)$ where:

$$f(v) = \frac{75}{v} + \frac{1}{3} + \frac{v}{1200}$$

Note that the distance 235 km. is not relevant. Once we find the optimum speed, it is optimum for all distances.

$$f'(v) = -\frac{75}{v^2} + \frac{1}{1200} = 0$$

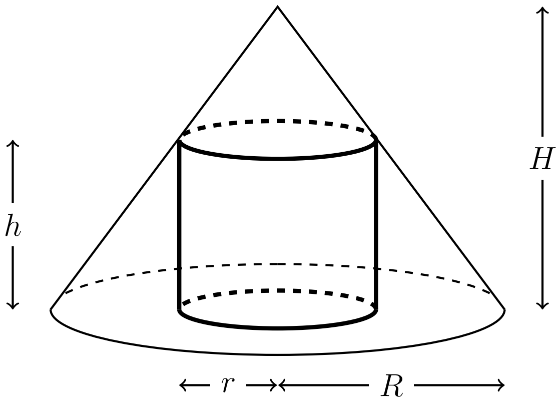
$$\Rightarrow \quad v^2 = 75 \cdot 1200 = 90\,000$$

$$\Rightarrow \quad v = 300$$

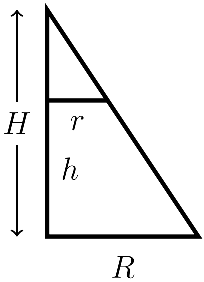
This value clearly gives the minimum, because:

$$\lim_{v \to 0} f = \lim_{v \to \infty} f = \infty.$$

Example 9–10: A cylinder is inscribed in a cone of radius R, height H. What is the maximum possible the volume of the cylinder?



Solution:



$V = \pi r^2 h.$

Using similar triangles we obtain:

$$\frac{H-h}{H} = \frac{r}{R}$$

$$\Rightarrow \quad h = H\left(1 - \frac{r}{R}\right)$$

$$V = \pi r^2 H\left(1 - \frac{r}{R}\right) = \pi H\left(r^2 - \frac{r^3}{R}\right)$$

$$\frac{dV}{dr} = \pi H\left(2r - \frac{3r^2}{R}\right) = 0$$

$$2r = \frac{3r^2}{R} \quad \Rightarrow \quad r = \frac{2R}{3} \quad \Rightarrow \quad h = \frac{H}{3}$$

Maximum Volume: $V = \dfrac{4}{27}\pi R^2 H.$

EXERCISES

Find the absolute maximum and minimum values of $f(x)$ on the given interval:

9–1) $f(x) = x^{\frac{2}{3}}$ on $\left[-2,\, 3\right]$

9–2) $f(x) = 10x\left(2 - \ln x\right)$ on $\left[1,\, e^2\right]$

9–3) $f(x) = 12 - x^2$ on $\left[2,\, 4\right]$

9–4) $f(x) = 12 - x^2$ on $\left[-2,\, 4\right]$

9–5) $f(x) = 3x^3 - 16x$ on $\left[-2,\, 1\right]$

9–6) $f(x) = x + \dfrac{9}{x}$ on $\left[1,\, 4\right]$

9–7) $f(x) = 3x^5 - 5x^3$ on $\left[-2,\, 2\right]$

9–8) $f(x) = |3x - 5|$ on $\left[0,\, 2\right]$

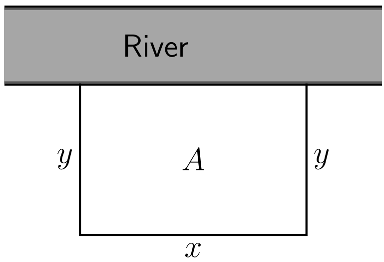
9–9) $f(x) = |x^2 + 6x - 7|$ on $\left[-8,\, 2\right]$

9–10) $f(x) = x\sqrt{1 - x^2}$ on $\left[-1,\, 1\right]$

9–11) $f(x) = e^{-x^2}$ on $\left[-1,\, 2\right]$

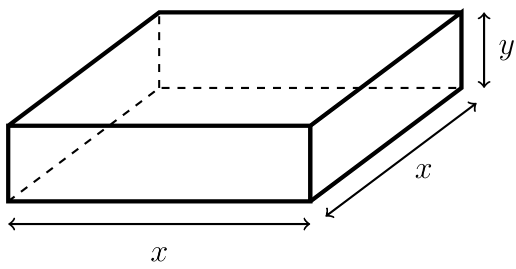
9–12) $f(x) = \dfrac{120}{\sqrt{x}}$ on $\left[16,\, 36\right]$

9–13) We will cover a rectangular area with a 36m-long fence. The area is near a river so we will only cover the three sides. Find the maximum possible area.



$$y + x + y = 36$$

9–14) An open top box has volume 75 cm^3 and is shaped as seen in the figure. Material for base costs $12\$/\text{cm}^2$ and material for sides costs $10\$/\text{cm}^2$. Find the dimensions x and y that give the minimum total cost.



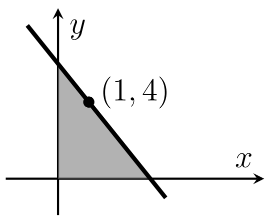
9–15) Find the dimensions of the right circular cylinder of the greatest volume if the surface area is 54π.

9–16) What is the maximum possible area of the rectangle with its base on the $x-$axis and its two upper vertices are on the graph of $y = 4 - x^2$?

9–17) Find the shortest distance between the point $(2, 0)$ and the curve $y = \sqrt{x}$.

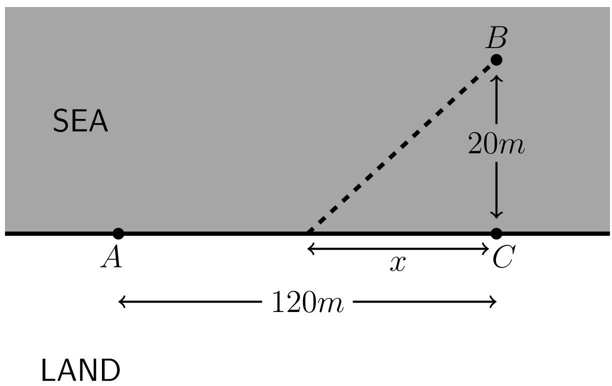
9–18) Find the point on the line $y = ax + b$ that is closest to origin.

9–19) We choose a line passing through the point $(1, 4)$ and find the area in the first quadrant bounded by the line and the coordinate axes. What line makes this area minimum?



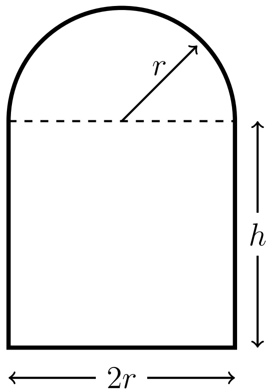
9–20) Two vertical poles are 21 meters apart. Their heights are 12m and 16m. A cable is stretched from the top of first pole to a point on the ground and then to the top of the second pole. Find the minimum possible length of the cable.

9–21) A swimmer is drowning on point B. You are at point A. You may run up to point C and then swim, or you may start swimming a distance x earlier. Assume your running speed is 5 m/s and your swimming speed is 3 m/s. What is the ideal x?



9–22) Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius R.

9–23) The window in the figure has fixed perimeter L. Find the dimensions that will maximize the amount of light. (i.e. area)



9–24) A coffee chain has 20 shops in a city. Average daily profit per shop is $\$3000$. Each new shop decreases the average profit of all shops by $\$100$. For example, if the company opens 3 new shops, average profit becomes $\$2700$.

What is the ideal number of shops, assuming the company wants to maximize total profit?

ANSWERS

9–1) Absolute Minimum: 0, Absolute Maximum: $\sqrt[3]{9}$.

9–2) Absolute Minimum: 0, Absolute Maximum: $10e$.

9–3) Absolute Minimum: -4, Absolute Maximum: 8.

9–4) Absolute Minimum: -4, Absolute Maximum: 12.

9–5) Absolute Minimum: -13, Absolute Maximum: $\dfrac{128}{9}$.

9–6) Absolute Minimum: 6, Absolute Maximum: 10.

9–7) Absolute Minimum: -56, Absolute Maximum: 56.

9–8) Absolute Minimum: 0, Absolute Maximum: 5.

9–9) Absolute Minimum: 0, Absolute Maximum: 16.

9–10) Absolute Minimum: $-\dfrac{1}{2}$, Absolute Maximum: $\dfrac{1}{2}$.

9–11) Absolute Minimum: $\dfrac{1}{e^2}$, Absolute Maximum: 1.

9–12) Absolute Minimum: 20, Absolute Maximum: 30.

9–13) $A = 162$

9–14) $x = 5, \quad y = 3$

9–15) $r = 3, \quad h = 6$

9–16) $\dfrac{32}{3\sqrt{3}}$

9–17) $\dfrac{\sqrt{7}}{2}$

9–18) $\left(\dfrac{-ab}{1+a^2}, \dfrac{b}{1+a^2}\right)$

9–19) $y = -4x + 8$

9–20) $35m$

9–21) $x = 15m$

9–22) $r = \dfrac{\sqrt{2}}{\sqrt{3}}\,R, \quad h = \dfrac{2}{\sqrt{3}}\,R$

9–23) $h = r = \dfrac{L}{\pi + 4}$

9–24) 25