

Math 111 – 10

Integrals

10.1 Indefinite Integrals

If f is the derivative of F , then F is the antiderivative of f .

$$F'(x) = f(x)$$

The collection of all antiderivatives of f is called the indefinite integral of f .

$$\int f(x) dx = F(x) + c$$

Here, c is an arbitrary constant.

Using the fact that integral and derivative are inverse operations, we obtain:

$$\int 1 dx = x + c$$

$$\int x dx = \frac{x^2}{2} + c$$

$$\int x^k dx = \frac{x^{k+1}}{k+1} + c, \quad k \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + c$$

$$\int e^x dx = e^x + c$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + c$$

Example 10–1: Evaluate the integral $\int \left(\frac{1}{x^3} - 2x + 4 \right) dx$

$$\begin{aligned}\textbf{Solution: } \int \left(\frac{1}{x^3} - 2x + 4 \right) dx &= \int \frac{dx}{x^3} - \int 2x \, dx + \int 4 \, dx \\&= \int x^{-3} \, dx - 2 \int x \, dx + \int 4 \, dx \\&= \frac{x^{-2}}{-2} - x^2 + 4x + c\end{aligned}$$

Example 10–2: Evaluate the integral $\int \frac{x^2 - 1}{x\sqrt{x}} \, dx$

$$\begin{aligned}\textbf{Solution: } \int \frac{x^2 - 1}{x\sqrt{x}} \, dx &= \int \frac{x^2}{x\sqrt{x}} \, dx - \int \frac{1}{x\sqrt{x}} \, dx \\&= \int x^{\frac{1}{2}} \, dx - \int x^{-\frac{3}{2}} \, dx \\&= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c \\&= \frac{2}{3}x\sqrt{x} + \frac{2}{\sqrt{x}} + c\end{aligned}$$

Example 10–3: Find a function $f(x)$ such that $f'(x) = 5e^x$ and $f(0) = 9$.

Solution: We have to integrate $5e^x$ to find $f(x)$:

$$\int 5e^x dx = 5e^x + c$$

Now, let's use the fact that $f(0) = 9$ to determine c :

$$5e^0 + c = 9 \Rightarrow 5 + c = 9 \Rightarrow c = 4$$

$$f(x) = 5e^x + 4.$$

Example 10–4: Find a function $f(x)$ such that $f''(x) = 4 - \frac{8}{x^2}$ and $f(1) = -15$, $f'(1) = 7$.

Solution: Let's integrate $4 - \frac{8}{x^2}$ to find $f'(x)$:

$$f'(x) = \int f''(x) dx = \int \left(4 - \frac{8}{x^2}\right) dx = 4x + \frac{8}{x} + c_1$$

Using $f'(1) = 7$ we find: $4 + 8 + c_1 = 7$

$$\Rightarrow c_1 = -5, \quad f'(x) = 4x + \frac{8}{x} - 5.$$

$$\begin{aligned} f(x) &= \int f'(x) dx \\ &= \int \left(4x + \frac{8}{x} - 5\right) dx = 2x^2 + 8 \ln|x| - 5x + c_2 \end{aligned}$$

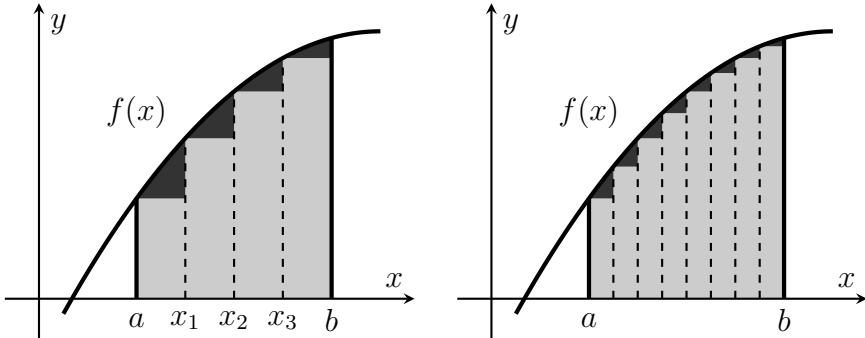
Using $f(1) = -15$ we find:

$$2 + 0 - 5 + c_2 = -15 \Rightarrow c_2 = -12.$$

$$\Rightarrow f(x) = 2x^2 + 8 \ln|x| - 5x - 12.$$

10.2 Definite Integrals

Area Under a Curve: We can approximate the area under the graph of f between $x = a$ and $x = b$ using rectangles.



This area is denoted by: $A = \int_a^b f(x)dx$.

Definite Integral Properties:

$$\int_a^a f(x)dx = 0$$

$$\int_a^b cf(x)dx = c \int_a^b f(x)dx$$

$$\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b f(x)dx = - \int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$(\text{Min } f) \cdot (b - a) \leq \int_a^b f(x)dx \leq (\text{Max } f) \cdot (b - a)$$

10.3 The Fundamental Theorem of Calculus

Let f be a continuous function on the interval $[a, b]$. If F is any anti-derivative of f , then

$$\int_a^b f(t) dt = F(b) - F(a)$$

We will use the notation $\int_a^b f(t) dt \Big|_a^b$ for $F(b) - F(a)$.

Example 10–5: Evaluate $\int_1^9 \frac{5}{\sqrt{x}} dx$

Solution:
$$\int_1^9 \frac{5}{\sqrt{x}} dx = \int_1^9 5x^{-\frac{1}{2}} dx$$

$$\begin{aligned} &= \frac{5x^{\frac{1}{2}}}{\frac{1}{2}} \Big|_1^9 \\ &= 5 \cdot 2 \cdot 9^{\frac{1}{2}} - 5 \cdot 2 \cdot 1^{\frac{1}{2}} \\ &= 30 - 10 = 20 \end{aligned}$$

Example 10–6: Evaluate $\int_3^7 \frac{dx}{x}$

Solution:
$$\begin{aligned} \int_3^7 \frac{dx}{x} &= \ln|x| \Big|_3^7 \\ &= \ln 7 - \ln 3 = \ln \frac{7}{3}. \end{aligned}$$

10.4 Substitution

Using the chain rule, we obtain:

$$\frac{d}{dx} F(u(x)) = \frac{dF(u)}{du} \cdot \frac{du(x)}{dx}$$

If we integrate both sides, we see that

$$\int f(u(x)) u'(x) dx = F(u(x)) + c$$

where $f = F'$, or more simply

$$\int f(u(x)) u'(x) dx = \int f(u) du$$

Here, the idea is to make a substitution that will simplify the given integral. For example, the choice $u = x^2 + 1$ simplifies the integral:

$$\int \frac{2x dx}{x^2 + 1} \rightarrow \int \frac{du}{u}$$

Example 10–7: Evaluate the integral $\int (x^4 + 1)^2 4x^3 dx$.

Solution: $u = x^4 + 1 \Rightarrow du = 4x^3 dx$

The new integral is:

$$I = \int u^2 du = \frac{u^3}{3} + c$$

But we have to express this in terms of the original variable:

$$I = \frac{(x^4 + 1)^3}{3} + c.$$

Example 10–8: Evaluate the integral $\int e^{3x^2} x \, dx$.

Solution: $u = 3x^2 \Rightarrow du = 6x \, dx$

$$\Rightarrow x \, dx = \frac{1}{6} du$$

$$\begin{aligned}\int e^{3x^2} x \, dx &= \int e^u \frac{1}{6} du \\ &= \frac{1}{6} e^u + c \\ &= \frac{e^{3x^2}}{6} + c.\end{aligned}$$

Example 10–9: Evaluate the integral $\int (x^3 + 6x^2)^7 (x^2 + 4x) \, dx$.

Solution: The substitution $u = x^3 + 6x^2$ gives:

$$du = (3x^2 + 12x) \, dx$$

$$\frac{1}{3} du = (x^2 + 4x) \, dx$$

Rewriting the integral in terms of u , we obtain:

$$\begin{aligned}\int (x^3 + 6x^2)^7 (x^2 + 4x) \, dx &= \frac{1}{3} \int u^7 du \\ &= \frac{u^8}{24} + c \\ &= \frac{(x^3 + 6x^2)^8}{24} + c\end{aligned}$$

10.5 Substitution in Definite Integrals

If u' is continuous on the interval $[a, b]$ and f is continuous on the range of u then

$$\int_a^b f(u(x)) u'(x) dx = \int_{u(a)}^{u(b)} f(u) du$$

Don't forget to transform the limits!

Example 10–10: Evaluate the integral $\int_1^2 \frac{x+2}{x^2+4x+1} dx$.

Solution: Use the substitution

$$\begin{aligned} u &= x^2 + 4x + 1 & \Rightarrow & \quad du = (2x+4) dx \\ && \Rightarrow & \quad \frac{1}{2} du = (x+2) dx \end{aligned}$$

The new integral limits are:

$$x = 1 \quad \Rightarrow \quad u = 6.$$

$$x = 2 \quad \Rightarrow \quad u = 13.$$

Rewriting the integral in terms of u , we obtain:

$$\begin{aligned} \int_1^2 \frac{x+2}{x^2+4x+1} dx &= \int_6^{13} \frac{\frac{1}{2} du}{u} \\ &= \frac{1}{2} \ln |u| \Big|_6^{13} \\ &= \frac{1}{2} \ln 13 - \frac{1}{2} \ln 6 \\ &= \frac{1}{2} \ln \frac{13}{6}. \end{aligned}$$

Example 10–11: Evaluate the definite integral $\int_0^1 8x(x^2 + 2)^3 dx$.

Solution: • Using $u = x^2 + 2$, $du = 2x dx$ and

$$x = 0 \Rightarrow u = 2,$$

$$x = 1 \Rightarrow u = 3.$$

we obtain:

$$I = \int_2^3 4u^3 du$$

$$= u^4 \Big|_2^3$$

$$= 81 - 16$$

$$= 65.$$

- Another idea is to evaluate it as an indefinite integral, rewrite u in terms of x and then use limits for x .

Once again, using $u = x^2 + 2$, $du = 2x dx$

$$\int 8x(x^2 + 2)^3 dx = \int 4u^3 du$$

$$= u^4 + c$$

$$= (x^2 + 2)^4 + c$$

$$\int_0^1 8x(x^2 + 2)^3 dx = (x^2 + 2)^4 \Big|_0^1$$

$$= 81 - 16$$

$$= 65.$$

Example 10–12: Evaluate $\int_{-4}^4 \frac{x}{\sqrt{5-x}} dx$.

Solution: Using $u = 5 - x$, $du = -dx$, $x = 5 - u$ and

$$x = -4 \Rightarrow u = 9, \quad x = 4 \Rightarrow u = 1,$$

$$\begin{aligned} \text{we obtain: } I &= - \int_9^1 \frac{5-u}{\sqrt{u}} du \\ &= 10u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}} \Big|_1^9 \\ &= (30 - 18) - \left(10 - \frac{2}{3}\right) \\ &= \frac{8}{3}. \end{aligned}$$

Example 10–13: Evaluate $\int_0^2 8e^{x^4/5}x^3 dx$.

Solution: Substitution: $u = \frac{x^4}{5}$, $du = \frac{4}{5}x^3 dx$.

New limits:

$$x = 0 \Rightarrow u = 0, \quad x = 2 \Rightarrow u = \frac{16}{5}$$

$$\begin{aligned} I &= 10 \int_0^{16/5} e^u du \\ &= 10e^u \Big|_0^{16/5} \\ &= 10(e^{16/5} - e^0) \\ &= 10(e^{16/5} - 1). \end{aligned}$$

EXERCISES

Evaluate the following indefinite integrals:

$$\mathbf{10-1)} \int (x^2 + 3x^5) dx$$

$$\mathbf{10-2)} \int (1 + 10x - 24x^7) dx$$

$$\mathbf{10-3)} \int \frac{1}{7x^4} dx$$

$$\mathbf{10-4)} \int -5e^x dx$$

$$\mathbf{10-5)} \int \frac{2}{x} dx$$

$$\mathbf{10-6)} \int (x^4 - x\sqrt{x} + 2x) dx$$

$$\mathbf{10-7)} \int (x^2 + 4)(x - 5) dx$$

$$\mathbf{10-8)} \int \frac{1}{\sqrt[3]{x^5}} dx$$

$$\mathbf{10-9)} \int \frac{2u^2 - 3u + 12}{u^2} du$$

$$\mathbf{10-10)} \int \frac{3}{e^{-z}} dz$$

Evaluate the following definite integrals:

$$\mathbf{10-11)} \quad \int_0^1 \sqrt{x^3} \, dx$$

$$\mathbf{10-12)} \quad \int_1^{27} x^{-1/3} \, dx$$

$$\mathbf{10-13)} \quad \int_1^3 \frac{5}{u} \, du$$

$$\mathbf{10-14)} \quad \int_2^3 (x^2 + 5x - 1) \, dx$$

$$\mathbf{10-15)} \quad \int_0^1 e^x \, dx$$

$$\mathbf{10-16)} \quad \int_{-1}^2 (1 + 4e^x) \, dx$$

$$\mathbf{10-17)} \quad \int_1^4 5t^{-2} \, dt$$

$$\mathbf{10-18)} \quad \int_1^9 \frac{1 - \sqrt{x}}{\sqrt{x}} \, dx$$

$$\mathbf{10-19)} \quad \int_1^{32} x^{-2/5} \, dx$$

$$\mathbf{10-20)} \quad \int_{-2}^{-1} \frac{1}{x^3} \, dx$$

Evaluate the following integrals: (Hint: Use substitution.)

$$\mathbf{10-21)} \int (1+x)^3 dx$$

$$\mathbf{10-22)} \int (x^2+1)^4 2x dx$$

$$\mathbf{10-23)} \int e^{8t} dt$$

$$\mathbf{10-24)} \int x^3 e^{-x^4} dx$$

$$\mathbf{10-25)} \int \frac{1}{x+4} dx$$

$$\mathbf{10-26)} \int \frac{3y}{(y^2-2)^4} dy$$

$$\mathbf{10-27)} \int \frac{2x^3+3x}{x^4+3x^2+1} dx$$

$$\mathbf{10-28)} \int e^{5x^3} 7x^2 dx$$

$$\mathbf{10-29)} \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

$$\mathbf{10-30)} \int x(3x^2+7)^5 dx$$

Evaluate the following integrals: (Hint: Use substitution.)

$$10-31) \int \sqrt{7x - 12} dx$$

$$10-32) \int \frac{1}{1 + 5z} dz$$

$$10-33) \int \frac{3}{(2 - x)^2} dx$$

$$10-34) \int \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} dx$$

$$10-35) \int 2y \sqrt{5 - 2y^2} dy$$

$$10-36) \int \frac{e^{1/x}}{x^2} dx$$

$$10-37) \int \frac{e^x + 3x^2}{e^x + x^3 + 1} dx$$

$$10-38) \int \frac{6z^2 + 8z + 3}{z + 1} dz$$

$$10-39) \int \frac{\ln x}{x} dx$$

$$10-40) \int \frac{1}{x(2 + \ln x)^3} dx$$

Evaluate the following definite integrals:

$$\mathbf{10-41)} \int_0^1 (x^2 + 1)^3 x \, dx$$

$$\mathbf{10-42)} \int_2^3 2e^{x^2} x \, dx$$

$$\mathbf{10-43)} \int_0^1 x^3 \sqrt[3]{x^4 + 1} \, dx$$

$$\mathbf{10-44)} \int_0^1 (1+t)^3 \, dt$$

$$\mathbf{10-45)} \int_1^2 \frac{\ln(t^2)}{t} \, dt$$

$$\mathbf{10-46)} \int_0^1 \frac{x+2x^3}{1+x^2+x^4} \, dx$$

$$\mathbf{10-47)} \int_0^1 e^{x^2+x} (2x+1) \, dx$$

$$\mathbf{10-48)} \int_0^{\ln 2} \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$$

$$\mathbf{10-49)} \int_0^{1/\sqrt{3}} (1+3t^2)^7 t \, dt$$

$$\mathbf{10-50)} \int_1^{e^5} \frac{1+2\ln x}{x} \, dx$$

Find the solution of the following differential equations:

$$\mathbf{10-51)} \quad \frac{dy}{dx} = 2, \quad y(0) = 9.$$

$$\mathbf{10-52)} \quad \frac{dy}{dx} = x\sqrt{x}, \quad y(1) = 1.$$

$$\mathbf{10-53)} \quad \frac{dy}{dx} = -2(x-2)^{-3} + 12, \quad y(3) = 37.$$

$$\mathbf{10-54)} \quad \frac{dy}{dx} = 24x^2, \quad y(0) = 4.$$

$$\mathbf{10-55)} \quad \frac{dy}{dx} = e^{-x}, \quad y(0) = 0.$$

$$\mathbf{10-56)} \quad \frac{dy}{dx} = \frac{1}{2}(1+x)^{-1/2} + 1, \quad y(3) = 5.$$

$$\mathbf{10-57)} \quad \frac{dy}{dx} = \frac{5}{(1-x)^{3/2}}, \quad y(0) = 15.$$

$$\mathbf{10-58)} \quad \frac{dy}{dx} = \frac{1}{x}, \quad y(1) = 4.$$

$$\mathbf{10-59)} \quad \frac{dy}{dx} = -6e^{-2x} - e^{-x}, \quad y(0) = 5.$$

$$\mathbf{10-60)} \quad \frac{dy}{dx} = \frac{2x}{1+x^2}, \quad y(0) = 0.$$

Answers

10-1) $\frac{x^3}{3} + \frac{x^6}{2} + c$

10-2) $x + 5x^2 - 3x^8 + c$

10-3) $-\frac{1}{21x^3} + c$

10-4) $-5e^x + c$

10-5) $2 \ln |x| + c$

10-6) $\frac{x^5}{5} - \frac{2x^{5/2}}{5} + x^2 + c$

10-7) $\frac{x^4}{4} - \frac{5x^3}{3} + 2x^2 - 20x + c$

10-8) $-\frac{3}{2}x^{-2/3} + c$

10-9) $2u - 3 \ln |u| - 12u^{-1} + c$

10-10) $3e^z + c$

$$\mathbf{10-11)} \quad \frac{2}{5}$$

$$\mathbf{10-12)} \quad 12$$

$$\mathbf{10-13)} \quad 5 \ln 3$$

$$\mathbf{10-14)} \quad \frac{107}{6}$$

$$\mathbf{10-15)} \quad e - 1$$

$$\mathbf{10-16)} \quad 3 + 4(e^2 - e^{-1})$$

$$\mathbf{10-17)} \quad \frac{15}{4}$$

$$\mathbf{10-18)} \quad -4$$

$$\mathbf{10-19)} \quad \frac{35}{3}$$

$$\mathbf{10-20)} \quad -\frac{3}{8}$$

$$\mathbf{10-21)} \quad \frac{(1+x)^4}{4} + c$$

$$\mathbf{10-22)} \quad \frac{(x^2+1)^5}{5} + c$$

$$\mathbf{10-23)} \quad \frac{e^{8t}}{8} + c$$

$$\mathbf{10-24)} \quad -\frac{e^{-x^4}}{4} + c$$

$$\mathbf{10-25)} \quad \ln|x+4| + c$$

$$\mathbf{10-26)} \quad -\frac{1}{2(y^2-2)^3} + c$$

$$\mathbf{10-27)} \quad \frac{1}{2} \ln|x^4 + 3x^2 + 1| + c$$

$$\mathbf{10-28)} \quad \frac{7}{15} e^{5x^3} + c$$

$$\mathbf{10-29)} \quad 2e^{\sqrt{x}} + c$$

$$\mathbf{10-30)} \quad \frac{(3x^2+7)^6}{36} + c$$

$$\mathbf{10-31)} \quad \frac{2}{21}(7x - 12)^{3/2} + c$$

$$\mathbf{10-32)} \quad \frac{1}{5} \ln |1 + 5z| + c$$

$$\mathbf{10-33)} \quad \frac{3}{2-x} + c$$

$$\mathbf{10-34)} \quad \frac{-2}{1+\sqrt{x}} + c$$

$$\mathbf{10-35)} \quad -\frac{1}{3}(5 - 2y^2)^{3/2} + c$$

$$\mathbf{10-36)} \quad -e^{1/x} + c$$

$$\mathbf{10-37)} \quad \ln |e^x + x^3 + 1| + c$$

$$\mathbf{10-38)} \quad 3z^2 + 2z + \ln |z + 1| + c$$

$$\mathbf{10-39)} \quad \frac{(\ln x)^2}{2} + c$$

$$\mathbf{10-40)} \quad \frac{-1}{2(2 + \ln x)^2} + c$$

$$\mathbf{10-41)} \quad \frac{15}{8}$$

$$\mathbf{10-42)} \quad e^9 - e^4$$

$$\mathbf{10-43)} \quad \frac{3}{16} (2^{4/3} - 1)$$

$$\mathbf{10-44)} \quad \frac{15}{4}$$

$$\mathbf{10-45)} \quad (\ln 2)^2$$

$$\mathbf{10-46)} \quad \frac{\ln 3}{2}$$

$$\mathbf{10-47)} \quad e^2 - 1$$

$$\mathbf{10-48)} \quad \ln \left(\frac{5}{4} \right)$$

$$\mathbf{10-49)} \quad \frac{85}{16}$$

$$\mathbf{10-50)} \quad 30$$

$$\mathbf{10-51)} \quad y = 2x + 9$$

$$\mathbf{10-52)} \quad y = \frac{2}{5}x^{5/2} + \frac{3}{5}$$

$$\mathbf{10-53)} \quad y = \frac{1}{(x - 2)^2} + 12x$$

$$\mathbf{10-54)} \quad y = 8x^3 + 4$$

$$\mathbf{10-55)} \quad y = 1 - e^{-x}$$

$$\mathbf{10-56)} \quad y = \sqrt{1+x} + x$$

$$\mathbf{10-57)} \quad y = \frac{10}{\sqrt{1-x}} + 5$$

$$\mathbf{10-58)} \quad y = \ln x + 4$$

$$\mathbf{10-59)} \quad y = 3e^{-2x} + e^{-x} + 1$$

$$\mathbf{10-60)} \quad y = \ln|1+x^2|$$