

# Math 111 – 11

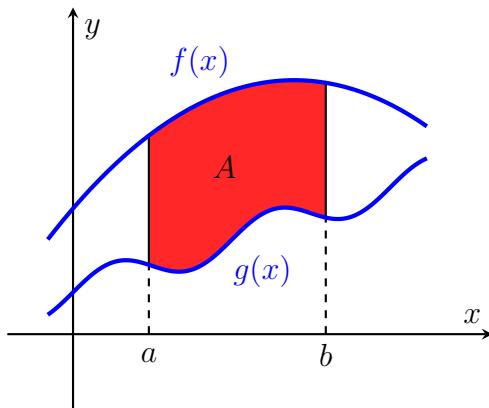
## Integral Techniques

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### 11.1 Area Between Curves

Let  $f$  and  $g$  be continuous with  $f(x) \geq g(x)$  on  $[a, b]$ . Then the area of the region bounded by the curves  $y = f(x)$ ,  $y = g(x)$  and the vertical lines  $x = a$ ,  $x = b$  is:

$$A = \int_a^b (f(x) - g(x)) dx$$

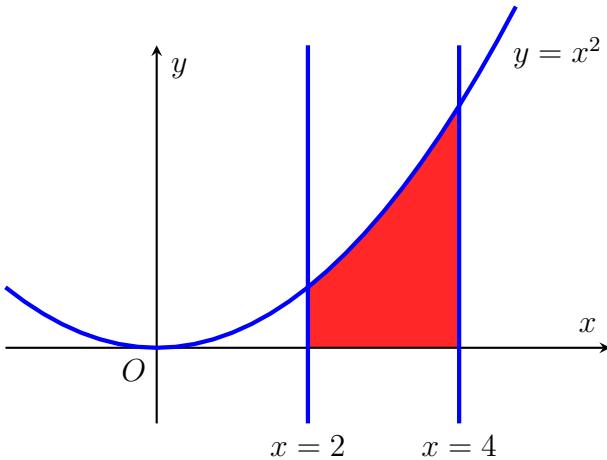


Note that we should integrate:

TOP function – BOTTOM function

**Example 11–1:** Find the area between the curve  $y = x^2$ , the lines  $x = 2$ ,  $x = 4$  and  $x$ -axis.

**Solution:** Let's sketch the graph first:

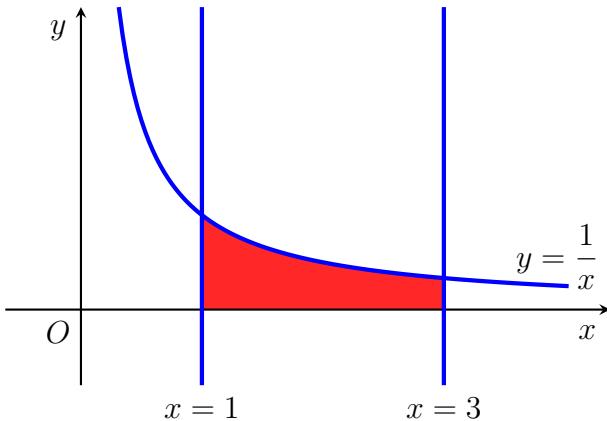


The integral that gives the shaded area is:

$$\begin{aligned} A &= \int_2^4 (x^2 - 0) dx \\ &= \frac{x^3}{3} \Big|_2^4 \\ &= \frac{4^3}{3} - \frac{2^3}{3} \\ &= \frac{56}{3} \end{aligned}$$

**Example 11–2:** Find the area between the curve  $y = \frac{1}{x}$ , the lines  $x = 1$ ,  $x = 3$  and  $x$ -axis.

**Solution:** Let's sketch the graph first:



The integral that gives the shaded area is:

$$A = \int_1^3 \left( \frac{1}{x} - 0 \right) dx$$

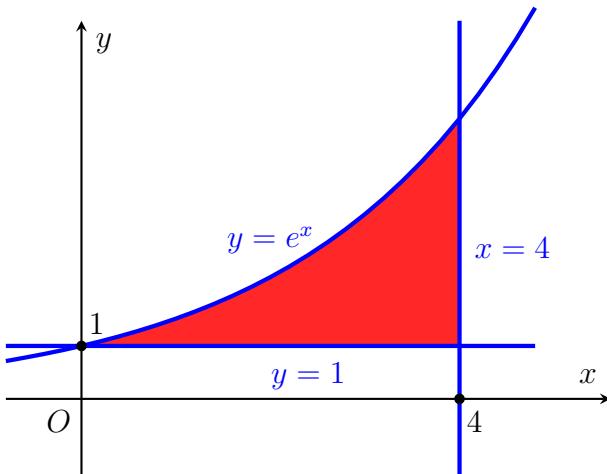
$$= \ln x \Big|_1^3$$

$$= \ln 3 - \ln 1$$

$$= \ln 3$$

**Example 11–3:** Find the area between the curve  $y = e^x$ , the lines  $y = 1$  and  $x = 4$ .

**Solution:** The region is between a curve and two lines. One is horizontal, the other is vertical.

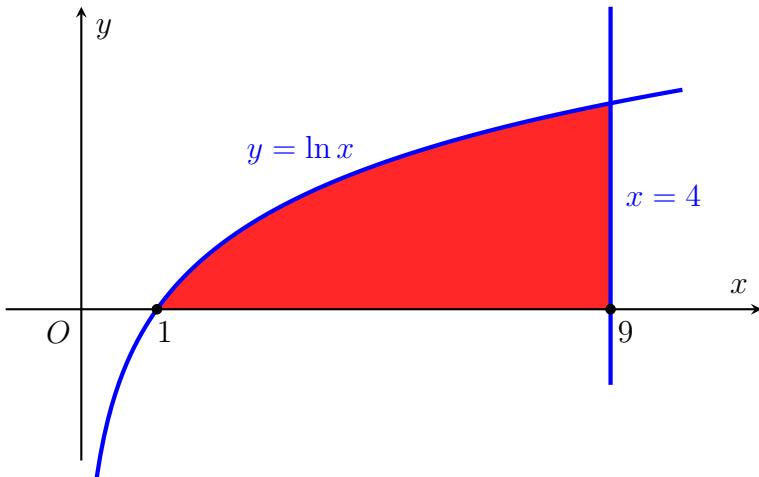


The area is:

$$\begin{aligned} A &= \int_0^4 (e^x - 1) dx \\ &= e^x - x \Big|_0^4 \\ &= (e^4 - 4) - (e^0 - 0) \\ &= e^4 - 5 \end{aligned}$$

**Example 11–4:** Find the area between the curve  $y = \ln x$ , the line  $x = 9$  and  $x$ -axis.

**Solution:** Remember that  $\ln 1 = 0$  and  $\ln 0$  is undefined.



The integral  $\int \ln x \, dx$  can be evaluated using integration by parts, where  $u = \ln x$  and  $dv = dx$ . The result is

$$\int \ln x \, dx = x \ln x - x + c$$

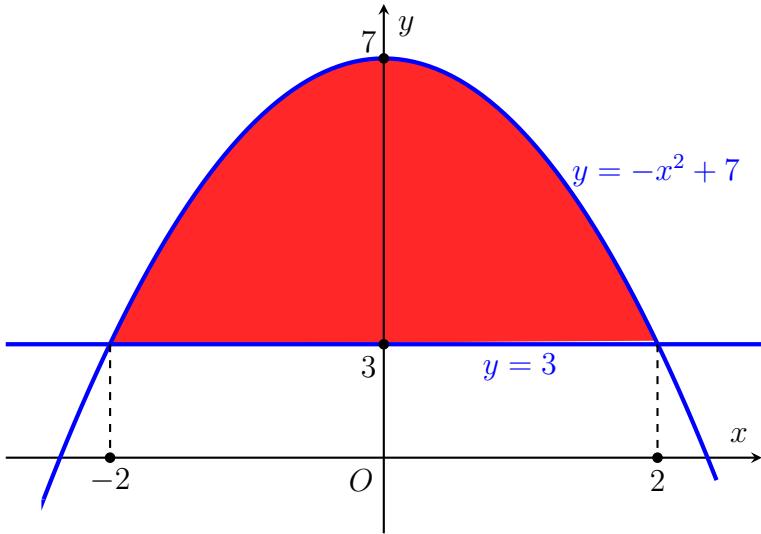
The area is:

$$\begin{aligned} A &= \int_1^9 \ln x \, dx \\ &= x \ln x - x \Big|_1^9 \\ &= (9 \ln 9 - 9) - (1 \ln 1 - 1) \\ &= 9 \ln 9 - 8 \end{aligned}$$

**Example 11–5:** Find the area between the curve  $y = -x^2 + 7$  and the line  $y = 3$ .

**Solution:** First, we have to find intersection points.

$$-x^2 + 7 = 3 \quad \Rightarrow \quad x = \pm 2$$



The area is:

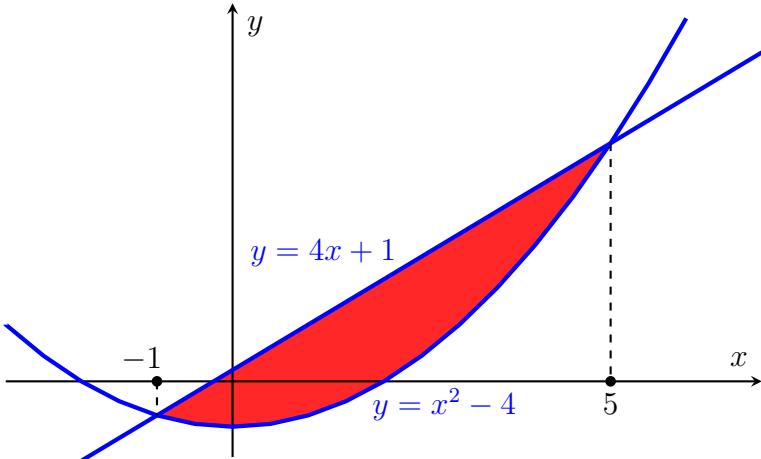
$$\begin{aligned} A &= \int_{-2}^2 (-x^2 + 7 - 3) dx \\ &= -\frac{x^3}{3} + 4x \Big|_{-2}^2 \\ &= \left(-\frac{8}{3} + 8\right) - \left(\frac{8}{3} - 8\right) \\ &= \frac{32}{3} \end{aligned}$$

**Example 11–6:** Find the area between the curve  $y = x^2 - 4$  and the line  $y = 4x + 1$ .

**Solution:** First, we have to find intersection points.

$$x^2 - 4 = 4x + 1 \quad \Rightarrow \quad x^2 - 4x - 5 = 0$$

$$(x + 1)(x - 5) = 0 \quad \Rightarrow \quad x = 5 \quad \text{or} \quad x = -1$$



The area is:

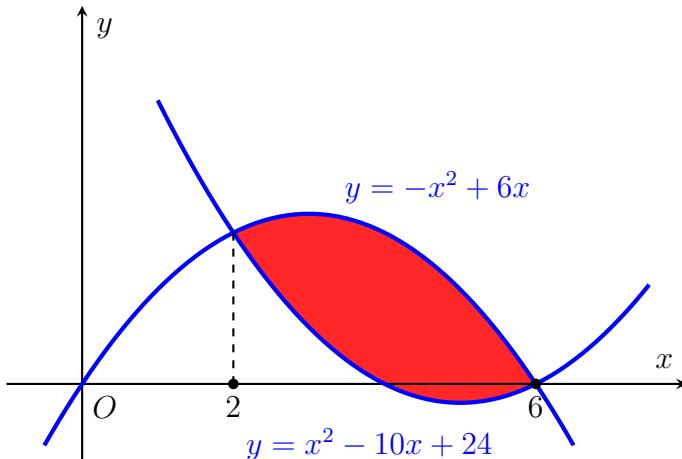
$$\begin{aligned} A &= \int_{-1}^5 (4x + 1 - x^2 + 4) dx \\ &= -\frac{x^3}{3} + 2x^2 + 5x \Big|_{-1}^5 \\ &= \left(-\frac{125}{3} + 50 + 25\right) - \left(\frac{1}{3} + 2 - 5\right) \\ &= 36 \end{aligned}$$

**Example 11–7:** Find the area between the curves  $y = x^2 - 10x + 24$  and  $y = -x^2 + 6x$ .

**Solution:** First, we have to find intersection points.

$$x^2 - 10x + 24 = -x^2 + 6x \Rightarrow 2x^2 - 16x + 24 = 0$$

$$2(x - 2)(x - 6) = 0 \Rightarrow x = 2 \text{ or } x = 6$$



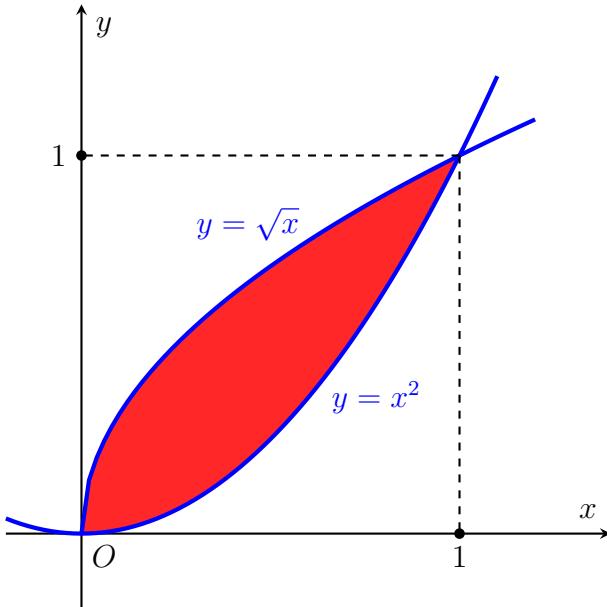
The area is:

$$\begin{aligned} A &= \int_2^6 \left[ (-x^2 + 6x) - (x^2 - 10x + 24) \right] dx \\ &= \int_2^6 \left[ -2x^2 + 16x - 24 \right] dx \\ &= -\frac{2x^3}{3} + 8x^2 - 24x \Big|_2^6 \\ &= (-144 + 288 + 144) - \left( -\frac{16}{3} + 32 - 48 \right) \\ &= \frac{64}{3} \end{aligned}$$

**Example 11–8:** Find the area between the curves  $y = x^2$  and  $y = \sqrt{x}$ .

**Solution:** First, we have to find intersection points.

$$x^2 = \sqrt{x} \Rightarrow x = 0 \text{ or } x = 1$$



The area is:

$$\begin{aligned} A &= \int_0^1 (\sqrt{x} - x^2) dx \\ &= \frac{2}{3} x^{3/2} - \frac{x^3}{3} \Big|_0^1 \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \frac{1}{3} \end{aligned}$$

## 11.2 Integration by Parts

$$\int u dv = uv - \int v du$$

**Example 11–9:** Evaluate  $\int xe^x dx$ .

**Solution:** Here,  $udv = xe^x dx$ .

$$u = x \quad \Rightarrow \quad du = dx$$

$$dv = e^x dx \quad \Rightarrow \quad v = e^x$$

Using the integration by parts formula, we obtain:

$$\begin{aligned}\int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + c.\end{aligned}$$

Note that if we start with the alternative choice

$$u = e^x \quad \Rightarrow \quad du = e^x dx$$

$$dv = x dx \quad \Rightarrow \quad v = \frac{x^2}{2}$$

we obtain

$$e^x \frac{x^2}{2} - \int e^x \frac{x^2}{2} dx$$

which is more complicated than the given integral.

**Example 11–10:** Evaluate the integral

$$\int (x+1)e^{3x} dx$$

**Solution:** We have to use integration by parts.

$$(x+1)e^{3x} dx = u dv$$

$$u = x+1 \quad \Rightarrow \quad du = dx$$

$$dv = e^{3x} dx \quad \Rightarrow \quad v = \frac{e^{3x}}{3}$$

$$\int (x+1)e^{3x} dx = (x+1) \frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} dx$$

$$= \frac{1}{3}(x+1)e^{3x} - \frac{1}{3} \int e^{3x} dx$$

$$= \frac{1}{3}(x+1)e^{3x} - \frac{e^{3x}}{9} + c$$

$$= \frac{1}{3}xe^{3x} + \frac{2}{9}e^{3x} + c$$

**Example 11–11:** Evaluate the integral  $\int x^3 \ln x dx$

**Solution:** We have to use integration by parts.

$$u = \ln x \quad \Rightarrow \quad du = \frac{dx}{x}$$

$$dv = x^3 dx \quad \Rightarrow \quad v = \frac{x^4}{4}$$

$$\int x^3 \ln x dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \frac{dx}{x}$$

$$= \frac{x^4 \ln x}{4} - \frac{1}{4} \int x^3 dx$$

$$= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + c.$$

**Example 11–12:** Evaluate  $\int \ln x dx$ .

**Solution:**  $u = \ln x \quad \Rightarrow \quad du = \frac{dx}{x}$

$$dv = dx \quad \Rightarrow \quad v = x$$

Using the integration by parts formula, we obtain:

$$\int \ln x dx = x \ln x - \int x \frac{dx}{x}$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + c.$$

## Definite Integrals using Integration by Parts:

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$$

Don't forget the limits for the  $uv$  term!

**Example 11–13:** Evaluate  $\int_1^e x \ln x \, dx$ .

**Solution:**  $u = \ln x \quad \Rightarrow \quad du = \frac{dx}{x}$

$$dv = x \, dx \quad \Rightarrow \quad v = \frac{x^2}{2}$$

Using the formula, we obtain:

$$\begin{aligned} \int_1^e x \ln x \, dx &= \frac{x^2}{2} \ln x \Big|_1^e - \int_1^e \frac{x^2}{2} \frac{dx}{x} \\ &= \frac{x^2}{2} \ln x \Big|_1^e - \frac{1}{2} \int_1^e x \, dx \\ &= \left( \frac{x^2}{2} \ln x - \frac{x^2}{4} \right) \Big|_1^e \\ &= \left( \frac{e^2}{2} \ln e - \frac{e^2}{4} \right) - \left( \frac{1}{2} \ln 1 - \frac{1}{4} \right) \\ &= \frac{e^2 + 1}{4} \end{aligned}$$

**Example 11–14:** Evaluate  $\int_1^e \frac{\ln x}{\sqrt{x}} dx$ .

**Solution:**  $u = \ln x \Rightarrow du = \frac{dx}{x}$

$$dv = \frac{dx}{\sqrt{x}} \Rightarrow v = 2\sqrt{x}$$

Using the formula, we obtain:

$$\begin{aligned}\int_1^e \frac{\ln x}{\sqrt{x}} dx &= 2\sqrt{x} \ln x \Big|_1^e - \int_1^e 2\sqrt{x} \frac{dx}{x} \\&= 2\sqrt{x} \ln x \Big|_1^e - \int_1^e \frac{2}{\sqrt{x}} dx \\&= \left(2\sqrt{x} \ln x - 4\sqrt{x}\right) \Big|_1^e \\&= \left(2\sqrt{e} \ln e - 4\sqrt{e}\right) - \left(2\sqrt{1} \ln 1 - 4\sqrt{1}\right) \\&= 4 - 2\sqrt{e}.\end{aligned}$$

## EXERCISES

Find the area of the region bounded by the given curves and lines:

**11–1)**  $y = 3x^2 + 2x$ ,  $x = 1$ ,  $x = 3$ ,  $x$  – axis.

**11–2)**  $y = -x^2 + 8x$ ,  $x = 0$ ,  $x = 4$ ,  $x$  – axis.

**11–3)**  $y = 12x^3$ ,  $x = 1$ ,  $x = 2$ ,  $x$  – axis.

**11–4)**  $y = 5 - x^4$ ,  $x = 0$ ,  $x = 1$ ,  $x$  – axis.

**11–5)**  $y = \frac{3}{x^2}$ ,  $x = 1$ ,  $x = 3$ ,  $x$  – axis.

**11–6)**  $y = \frac{1}{x}$ ,  $x = 2$ ,  $x = 10$ ,  $x$  – axis.

**11–7)**  $y = \frac{8}{(x+1)^2}$ ,  $x = 0$ ,  $x = 3$ ,  $x$  – axis.

**11–8)**  $y = e^x$ ,  $x = 2$ ,  $x = 5$ ,  $x$  – axis.

**11–9)**  $y = e^{-x}$ ,  $x = 0$ ,  $x = 3$ ,  $x$  – axis.

**11–10)**  $y = e^{2x}$ ,  $x = -2$ ,  $x = 1$ ,  $x$  – axis.

**11–11)**  $y = 3\sqrt{x}$ ,  $x = 1$ ,  $x = 9$ ,  $x$  – axis.

**11–12)**  $y = \sqrt{x}$ ,  $x = 0$ ,  $x = 6$ ,  $x$  – axis.

**11–13)**  $y = \ln x$ ,  $x = 2$ ,  $x = 4$ ,  $x$  – axis.

Find the area of the region bounded by the given curves and lines:  
(Hint: First, find the intersection points.)

**11–14)**  $y = x^2 + 1$  and  $y = 2x + 1$ .

**11–15)**  $y = -x^2 + 5x$  and  $y = 6$ .

**11–16)**  $y = x^2 - 6$  and  $y = x$ .

**11–17)**  $y = -x^2 + 25$  and  $y = -2x + 10$ .

**11–18)**  $y = 12x^2 - 3$  and  $x$  – axis.

**11–19)**  $y = 6x - 2x^2$  and  $x$  – axis.

**11–20)**  $y = x^2 - 2x - 6$  and  $y = -x^2 + 2x$ .

**11–21)**  $y = x^2$  and  $y = 6 - 5x^2$ .

**11–22)**  $y = e^x$ ,  $y = e^{-x}$  and  $x = 1$ .

**11–23)**  $y = e^{-x}$ ,  $y = 1$  and  $x = 1$ .

**11–24)**  $y = 12\sqrt{x}$  and  $y = 3x$ .

**11–25)**  $y = \frac{5}{x}$  and  $y = 6 - x$ .

Evaluate the following integrals. (Hint: Use integration by parts)

$$\mathbf{11-26)} \quad \int \ln x \, dx$$

$$\mathbf{11-27)} \quad \int x^2 e^x \, dx$$

$$\mathbf{11-28)} \quad \int x^2 \ln x \, dx$$

$$\mathbf{11-29)} \quad \int \frac{\ln x}{\sqrt{x}} \, dx$$

$$\mathbf{11-30)} \quad \int x e^{ax} \, dx$$

$$\mathbf{11-31)} \quad \int_0^2 x e^{-x} \, dx$$

$$\mathbf{11-32)} \quad \int_1^2 x \ln x \, dx$$

$$\mathbf{11-33)} \quad \int_2^5 \ln(4x) \, dx$$

$$\mathbf{11-34)} \quad \int_1^3 4x e^{2x} \, dx$$

$$\mathbf{11-35)} \quad \int_1^8 \frac{\ln x}{x^{1/3}} \, dx$$

## ANSWERS

**11–1)** 34

**11–2)**  $\frac{128}{3}$

**11–3)** 45

**11–4)**  $\frac{24}{5}$

**11–5)** 2

**11–6)**  $\ln 5$

**11–7)** 6

**11–8)**  $e^5 - e^2$

**11–9)**  $1 - \frac{1}{e^3}$

**11–10)**  $\frac{e^2 - e^{-4}}{2}$

**11–11)** 52

**11–12)**  $4\sqrt{6}$

**11–13)**  $6 \ln 2 - 2$

$$\mathbf{11-14)} \quad \frac{4}{3}$$

$$\mathbf{11-15)} \quad \frac{1}{6}$$

$$\mathbf{11-16)} \quad \frac{125}{6}$$

$$\mathbf{11-17)} \quad \frac{256}{3}$$

$$\mathbf{11-18)} \quad 2$$

$$\mathbf{11-19)} \quad 9$$

$$\mathbf{11-20)} \quad \frac{64}{3}$$

$$\mathbf{11-21)} \quad 8$$

$$\mathbf{11-22)} \quad e + \frac{1}{e} - 2$$

$$\mathbf{11-23)} \quad \frac{1}{e}$$

$$\mathbf{11-24)} \quad 128$$

$$\mathbf{11-25)} \quad 12 - 5 \ln 5$$

$$\mathbf{11-26)} \quad x \ln x - x + c$$

$$\mathbf{11-27)} \quad x^2 e^x - 2x e^x + 2e^x + c$$

$$\mathbf{11-28)} \quad \frac{x^3}{3} \left( \ln x - \frac{1}{3} \right) + c$$

$$\mathbf{11-29)} \quad 2\sqrt{x} \ln x - 4\sqrt{x} + c$$

$$\mathbf{11-30)} \quad \frac{x e^{ax}}{a} - \frac{e^{ax}}{a^2} + c$$

$$\mathbf{11-31)} \quad 1 - 3e^{-2}$$

$$\mathbf{11-32)} \quad 2 \ln 2 - 3/4$$

$$\mathbf{11-33)} \quad 3 \ln 4 + 5 \ln 5 - 2 \ln 2 - 3$$

$$\mathbf{11-34)} \quad 5e^6 - e^2$$

$$\mathbf{11-35)} \quad 18 \ln 2 - \frac{27}{4}$$