Lecture Notes On

Mathematics for Business, Economics and Social Sciences

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Contents

Chapter 1

Functions

Equations: We can perfom the same operation on both sides of an equality:

$$\begin{aligned} 8x - 2 &= 5x + 7 && \text{Add } 2 \text{ to both sides} \\ 8x &= 5x + 9 && \text{Subtract } 5x \text{ from both sides} \\ 3x &= 9 && \text{Multiply both sides by } \tfrac{1}{3} \\ x &= 3 \end{aligned}$$

Example 1–1: Solve the equation $\dfrac{2x}{2x+5} = \dfrac{3}{4}$

Solution:

$$\begin{aligned} \frac{2x}{2x+5} &= \frac{3}{4} \\ 8x &= 6x + 15 \\ 2x &= 15 \\ x &= \frac{15}{2} \end{aligned}$$

Example 1–2: Solve the equation $\left|3x - 12\right| = 27$

Solution: Using the definition of absolute value, we get

$$\begin{aligned} 3x - 12 &= 27 && \text{or} && -3x + 12 &= 27 \\ 3x &= 39 && \text{or} && -3x &= 15 \\ x &= 13 && \text{or} && x &= -5 \end{aligned}$$

Intervals:

- Closed interval: $\big[a,\, b\big] = \{x : a \leqslant x \leqslant b\}$
- Open interval: $\big(a,\, b\big) = \{x : a < x < b\}$
- Half-open interval: $\big(a,\, b\big] = \{x : a < x \leqslant b\}$
- Unbounded interval: $\big(a,\, \infty\big) = \{x : a < x\}$

We will use $\mathbb{R}$ to denote all real numbers, in other words the interval $\big(-\infty,\, \infty\big)$.

Inequalities: Inequalities are similar to equations. We can add the same quantity to both sides, but if we multiply by a negative number, the direction of the inequality is reversed.

Example 1–3: Solve the inequality $7x - 5 \leqslant 30$.

Solution:

$$\begin{aligned} 7x - 5 &\leqslant 23 \\ 7x &\leqslant 28 \\ x &\leqslant 4 \\ x &\in \big(-\infty, 4\big] \end{aligned}$$

Example 1–4: Solve the inequality $\big|x + 10\big| < 11$.

Solution:

$$\begin{array}{ccccc} -11 & < & x+10 & < & 11 \\ -21 & < & x & < & 1 \end{array}$$

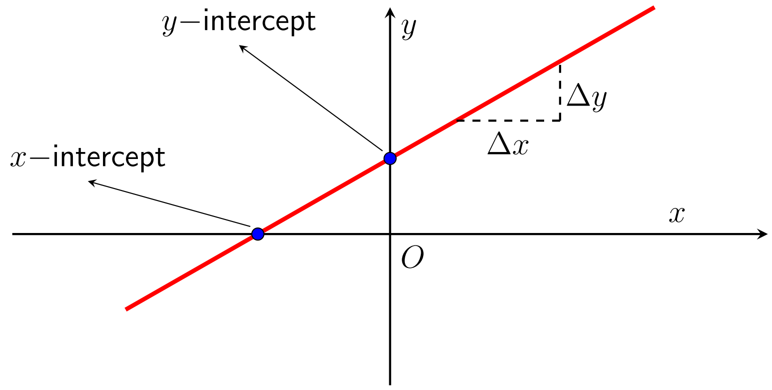
$$x \in \big(-21, 1\big)$$

Example 1–5: Solve the inequality $\big|x + 10\big| > 11$.

Solution:

$$\begin{array}{rclcrcl} x+10 & > & 11 & \text{or} & x+10 & < & -11 \\ x & > & 1 & \text{or} & x & < & -21 \\ x & \in & \big(-\infty, -21\big) & \text{or} & x & \in & \big(1, \infty\big) \end{array}$$

Lines on the Plane:



Slope of a line is:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$x = 0$ gives the $y-$intercept and $y = 0$ gives the $x-$intercept.

- Slope-intercept equation: $y = mx + n$.
- Point-slope equation: $y - y_1 = m(x - x_1)$.

If we are given two points on a line or one point and the slope, we can find the equation of the line.

- If two lines are parallel: $m_1 = m_2$.
- If two lines are perpendicular: $m_1 \cdot m_2 = -1$.

The equation $x = c$ gives a vertical line and $y = c$ gives a horizontal line.

Example 1–6: Find the equation of the line passing through the points $(2,9)$ and $(4,13)$.

Solution: Let's find the slope first: $m = \dfrac{13-9}{4-2} = 2$

Now, let's use the point-slope form of a line equation using the point $(2,9)$:

$(y-9) = 2(x-2)$

$y = 2x + 5$

If we use $(4,13)$, we will obtain the same result:

$(y-13) = 2(x-4)$

$y = 2x + 5$

Example 1–7: Find the equation of the line passing through the point $(2,4)$ and parallel to the line $3x + 5y = 1$.

Solution: If we rewrite the line equation as: $y = -\dfrac{3}{5}x + \dfrac{1}{5}$

we see that $m = -\dfrac{3}{5}$.

Therefore: $y - 4 = -\dfrac{3}{5}(x-2)$

$y = -\dfrac{3}{5}x + \dfrac{26}{5}$

or $3x + 5y = 26$.

Example 1–8: Find the equation of the line passing through the points $(24,0)$ and $(8,-6)$.

Solution: The slope is:

$$m = \frac{-6-0}{8-24} = \frac{-6}{-16} = \frac{3}{8}$$

Using point-slope equation, we find:

$y - 0 = \dfrac{3}{8}(x-24) \quad \Rightarrow \quad y = \dfrac{3}{8}x - 9$

In other words: $3x - 8y = 72$.

Example 1–9: Find the equation of the line passing through origin and parallel to the line $2y - 8x - 12 = 0$.

Solution: If we rewrite the line equation as:

$y = 4x + 6$,

we see that $m = 4$. Therefore:

$y - 0 = 4(x-0)$

$y = 4x$.

Note that a line through origin has zero intercept.

Function: A function f defined on a set D of real numbers is a rule that assigns to each number x in D exactly one real number, denoted by $f(x)$.
For example:

$$\begin{aligned} f(x) &= x^2 \\ f(x) &= 7x+2 \\ f(x) &= |x| \\ f(x) &= \frac{1}{x} \end{aligned}$$

Domain: The set D of all numbers for which $f(x)$ is defined is called the domain of the function f.

For example, consider the function $f(x) = 4x^2 + 5$.

There's no x value where $f(x)$ is undefined so its domain is $\mathbb{R}$.

Range: The set of all values of $f(x)$ is called the range of f.

$$\begin{aligned} x^2 &\geqslant 0 \\ 4x^2 &\geqslant 0 \\ 4x^2+5 &\geqslant 5 \end{aligned}$$

So range of $f(x) = 4x^2 + 5$ is: $\big[5,\,\infty\big)$.

The range and domain of a linear function $f(x) = ax + b$ is $\mathbb{R}$.

Example 1–10: Find the domain of the function

$$f(x) = \frac{1}{x+8}$$

Solution: Division by zero is undefined, in other words, it is not possible to evaluate this function at the point $x = -8$.

Therefore the domain is: $\mathbb{R} \setminus \{-8\}$.

We can also write this as: $\big(-\infty, -8\big) \cup \big(-8,\,\infty\big)$

Example 1–11: Find the domain of the function

$$f(x) = \sqrt{x-4}$$

Solution: Square root of a negative number is undefined. (In this course, we are not using complex numbers.) Therefore

$$x - 4 \geqslant 0 \quad \Rightarrow \quad x \geqslant 4$$

In other words, domain is: $\big[4,\,\infty\big)$

Example 1–12: Find the domain of the function

$$f(x) = \frac{1}{\sqrt{x-4}}$$

Solution: This is similar to previous exercise, but the function is not defined at $x = 4$. Therefore, the domain is: $\big(4,\,\infty\big)$

EXERCISES

Perform the following operations. Transform and simplify the result.

1–1) $(2^3)^2$

1–2) $\left(\dfrac{1}{16}\right)^{3/4}$

1–3) $72^{1/2}$

1–4) $\sqrt[3]{-125}$

1–5) $\sqrt[3]{\dfrac{8}{1000}}$

1–6) $\sqrt{\dfrac{48}{49}}$

1–7) $(a+b)^2$

1–8) $(a+b)(a-b)$

1–9) $\dfrac{1}{\sqrt{5}-\sqrt{3}}$

1–10) $\dfrac{12}{\sqrt{7}-1} - \dfrac{12}{\sqrt{7}+1}$

Perform the following operations. Transform and simplify the result.

1–11) $\sqrt{x^2\sqrt{x}}$

1–12) $\sqrt{x^3y}\,\sqrt{64xy^9}$

1–13) x^3-1

1–14) $\left(\sqrt{x^2+4}+3\right)\left(\sqrt{x^2+4}-3\right)$

1–15) x^4-100y^4

1–16) $\left(\dfrac{x^2\,y^{1/2}}{x^{2/3}\,y^{1/6}}\right)^3$

1–17) $(3a-2b)^2$

1–18) $(a+b)^3$

1–19) $\dfrac{2x}{x^2-4} + \dfrac{5}{x+2}$

1–20) $1 - \dfrac{1}{1+\dfrac{1}{x}}$

Solve the following equations and inequalities:

1–21) $3(x+7)-2(3x-4)=14$

1–22) $\frac{x}{3}-\frac{x}{5}=\frac{7}{30}$

1–23) $\sqrt{x^2+16}=5$

1–24) $|x-2|=12$

1–25) $\left|x-7\right|<8$

1–26) $\left|2x+6\right|\leqslant 4$

1–27) $\left|5x-10\right|>15$

1–28) $\left|12-7x\right|\geqslant 1$

1–29) $\left|x^2-5\right|<2$

1–30) $\left|x^2-5\right|<10$

Find the equations of the following lines:

1–31) Passes through origin and has slope $m=\frac{1}{5}$.

1–32) Passes through the point $(-2,6)$ and has slope $m=3$.

1–33) Passes through the points $(-8,2)$ and $(-1,-2)$.

1–34) Passes through $(0,-3)$ and parallel to the line $10y-5x=99$.

1–35) Passes through $(9,12)$ and perpendicular to the line $2x+5y=60$.

Find the domain and range of the following functions:

1–36) $f(x)=\sqrt{10-x}$

1–37) $f(x)=x^2+12x+35$

1–38) $f(x)=8x-x^2$

1–39) $f(x)=\frac{1}{x^2-6x+9}$

1–40) $f(x)=\frac{3}{x-7}$

ANSWERS

1–1) $2^3 \cdot 2^3 = 2^6 = 64$

1–2) $\left(2^{-4}\right)^{3/4} = 2^{-3} = \dfrac{1}{8}$

1–3) $\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$

1–4) $\left[(-5)^3\right]^{1/3} = -5$

1–5) $\dfrac{\sqrt[3]{8}}{\sqrt[3]{1000}} = \dfrac{2}{10} = 0.2$

1–6) $\dfrac{4\sqrt{3}}{7}$

1–7) $a^2 + 2ab + b^2$

1–8) $a^2 - b^2$

1–9) $\dfrac{\sqrt{5} + \sqrt{3}}{2}$

1–10) 4

1–11) $x^{5/4}$

1–12) $8x^2y^5$

1–13) $(x-1)(x^2+x+1)$

1–14) $x^2 - 5$

1–15) $(x^2 - 10y^2)(x^2 + 10y^2)$

1–16) x^4y

1–17) $9a^2 - 12ab + 4b^2$

1–18) $a^3 + 3a^2b + 3ab^2 + b^3$

1–19) $\dfrac{7x - 10}{x^2 - 4}$

1–20) $\dfrac{1}{x+1}$

1–21) $x = 5$

1–22) $x = \dfrac{7}{4}$

1–23) $x = \pm 3$

1–24) $x = 14$ or $x = -10$

1–25) $-1 < x < 15$

1–26) $-5 \leqslant x \leqslant -1$

1–27) $x < -1$ or $x > 5$

1–28) $x \leqslant \dfrac{11}{7}$ or $x \geqslant \dfrac{13}{7}$

1–29) $\sqrt{3} < x < \sqrt{7}$ or $-\sqrt{7} < x < -\sqrt{3}$

1–30) $-\sqrt{15} < x < \sqrt{15}$

1–31) $y = \dfrac{1}{5}x$

1–32) $y = 3x + 12$

1–33) $4x + 7y + 18 = 0$

1–34) $x - 2y = 6$

1–35) $5x - 2y = 21$

1–36) Domain: $\big(-\infty, 10\big]$, range: $\big[0, \infty\big)$.

1–37) Domain: $\mathbb{R}$, range: $\big[-1, \infty\big)$.

1–38) Domain: $\mathbb{R}$, range: $\big(-\infty, 16\big)$.

1–39) Domain: $\mathbb{R} \setminus \{3\}$, range: $\big(0, \infty\big)$.

1–40) Domain: $\mathbb{R} \setminus \{7\}$, range: $\big(-\infty, 0\big) \cup \big(0, \infty\big)$.

Chapter 2

Parabolas

Quadratic Equations: The solution of the equation

$$ax^2 + bx + c = 0$$

is:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} \quad \text{where} \quad \Delta = b^2 - 4ac$$

Here, we assume $a \neq 0$.

- If $\Delta > 0$, there are two distinct solutions.
- If $\Delta = 0$, there is a single solution.
- If $\Delta < 0$, there is no real solution.

(In this course, we only consider real numbers)

Example 2–1: Solve the equation $x^2 - 6x - 7 = 0$.

Solution: We can factor this equation as: $(x - 7)(x + 1) = 0$

Therefore $x - 7 = 0$ or $x + 1 = 0$.

In other words, $x = 7$ or $x = -1$.

Alternatively, we can use the formula to obtain the same result. Note that

$$a = 1,\ b = -6 \quad \text{and} \quad c = -7$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{6 \pm \sqrt{36 + 28}}{2} \\ &= \frac{6 \pm 8}{2} \end{aligned}$$

So $x = 7$ or $x = -1$.

Example 2–2: Solve $8x^2 - 6x - 5 = 0$.

Solution: Using the formula, we obtain:

$$x = \frac{6 \pm \sqrt{36 + 160}}{16} = \frac{6 \pm 14}{16}$$

So $x = \dfrac{5}{4}$ or $x = -\dfrac{1}{2}$.

Alternatively, we can see directly that

$(4x - 5)(2x + 1) = 0$, but this is not easy.

Example 2–3: Solve $9x^2 - 12x + 4 = 0$.

Solution: If we can see that this is a full square

$(3x - 2)^2 = 0$ we obtain $x = \dfrac{2}{3}$ easily.

Alternatively, $\Delta = (-12)^2 - 4 \cdot 9 \cdot 4 = 0$

(There is only one solution)

Example 2–4: Solve $3x^2 + 6x + 4 = 0$.

Solution:
$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= 36 - 48 \\ &= -12 \end{aligned}$$

$\Delta < 0 \quad \Rightarrow \quad$ There is no solution.

Quadratic Functions:

A function of the form

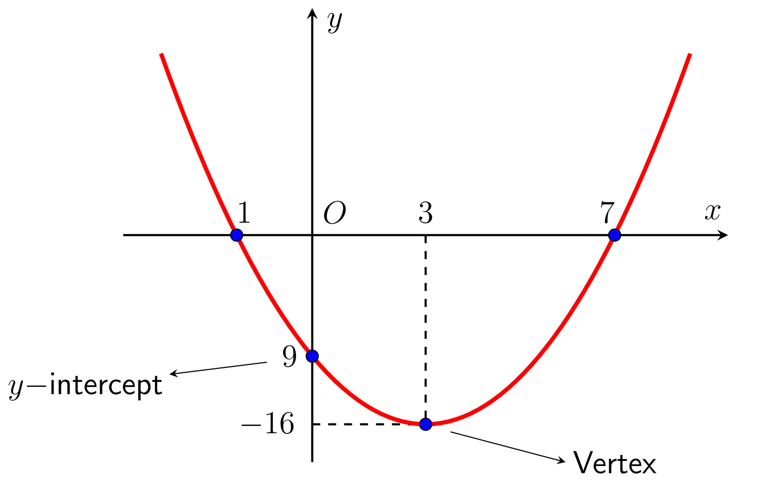
$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

is called a quadratic function. The graph of a quadratic function is a parabola.

- If $a > 0$, the arms of the parabola open upward.
- If $a < 0$, the arms of the parabola open downward.

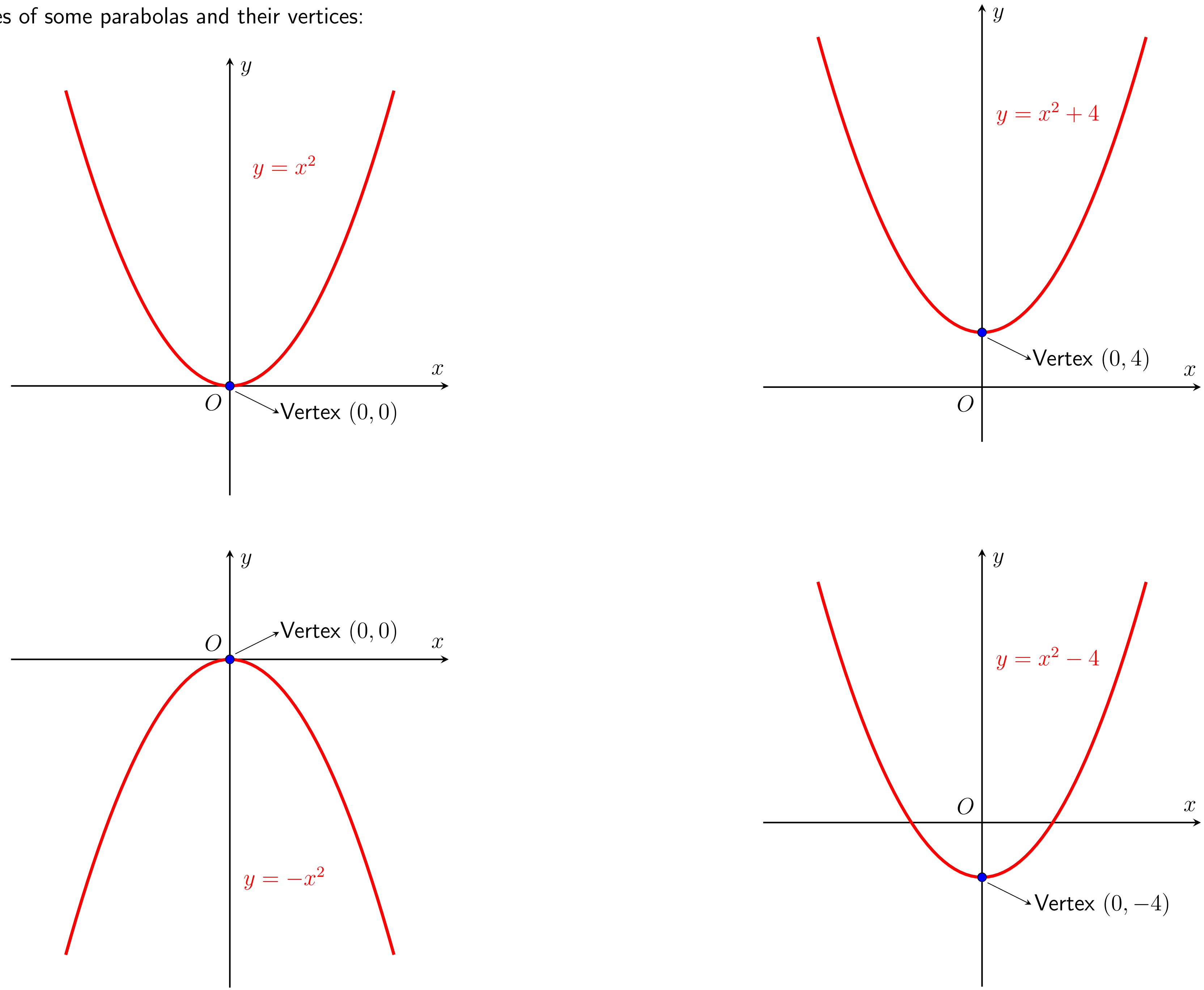
The vertex of the parabola is maximum or minimum point.

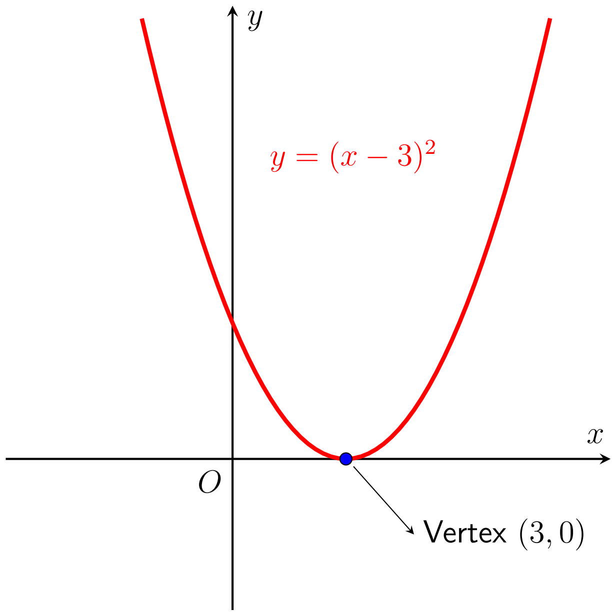
The $x-$coordinate of the vertex is $-\dfrac{b}{2a}$ and the $y-$coordinate is $f\left(-\dfrac{b}{2a}\right)$. An example is:

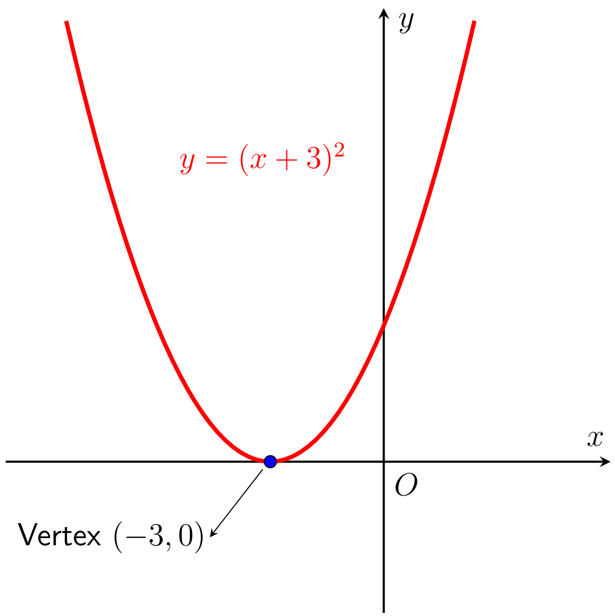


The graph of $y = x^2 - 6x - 7$

Examples of some parabolas and their vertices:







Example 2–5: Sketch the graph of $f(x)=x^2-10x+16$.

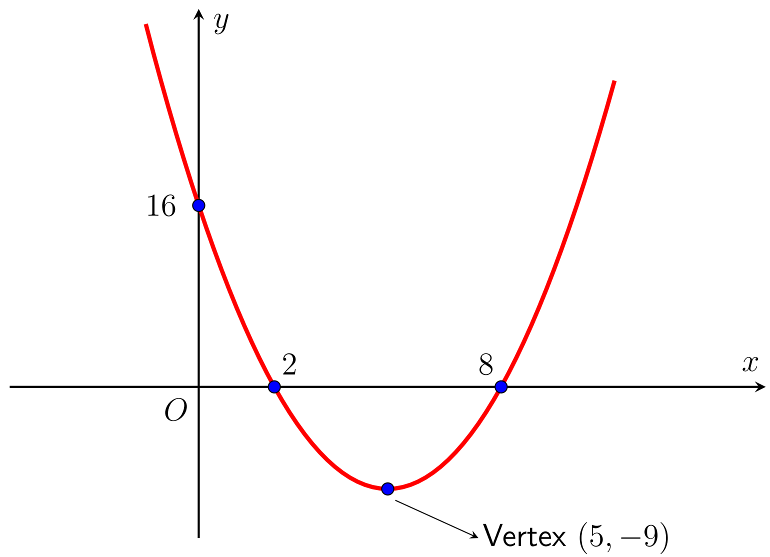
Solution: $y-$intercept: $x=0 \quad \Rightarrow \quad y=16$

Roots: $x^2-10x+16=0 \quad \Rightarrow \quad x=2$ or $x=8$.

Vertex: $-\dfrac{b}{2a}=\dfrac{10}{2}=5, \quad f(5)=-9.$

The coordinates of the vertex is $(5,-9)$.

$a>0 \quad \Rightarrow$ arms open upward. The graph is:



We can obtain the same graph by writing the given function in the form:

$$f(x)=(x-5)^2-9$$

EXERCISES

Solve the following quadratic equations:

2–1) $x^2 - 5x - 24 = 0$

2–2) $2x^2 + 9x - 5 = 0$

2–3) $6x^2 - 7x + 2 = 0$

2–4) $49x^2 - 14x + 1 = 0$

2–5) $4x^2 + 6x + 3 = 0$

2–6) $x^2 - 17x = 0$

2–7) $4x^2 - 20x + 25 = 0$

2–8) $x^2 - 4x + 5 = 0$

2–9) $x^2 - \dfrac{10}{3}x + 1 = 0$

2–10) $x^2 - 2x - 1 = 0$

Find the vertex and $x-$ and $y-$ intercepts of the following parabolas. Sketch their graphs:

2–11) $y = x^2 - 6x$

2–12) $y = -x^2 + 12$

2–13) $y = x^2 - 4x - 21$

2–14) $y = -x^2 + 3x + 4$

2–15) $y = x^2 + 10x + 25$

2–16) $y = 4x^2 - 8x + 3$

2–17) $y = 5x^2 + 15$

2–18) $y = -(x - 4)^2$

2–19) $y = x^2 - 4x + 5$

2–20) $y = -3x^2 + 60x - 450$

ANSWERS

2–1) $x_1 = 8, \quad x_2 = -3.$

2–2) $x_1 = \dfrac{1}{2}, \quad x_2 = -5.$

2–3) $x_1 = \dfrac{1}{2}, \quad x_2 = \dfrac{2}{3}.$

2–4) $x_1 = \dfrac{1}{7}$. (double root.)

2–5) There is no solution.

2–6) $x_1 = 0, \quad x_2 = 17.$

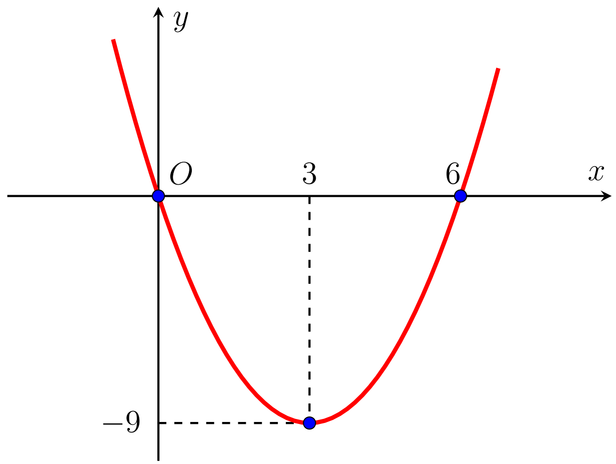
2–7) $x_1 = \dfrac{5}{2}$. (double root.)

2–8) There is no solution.

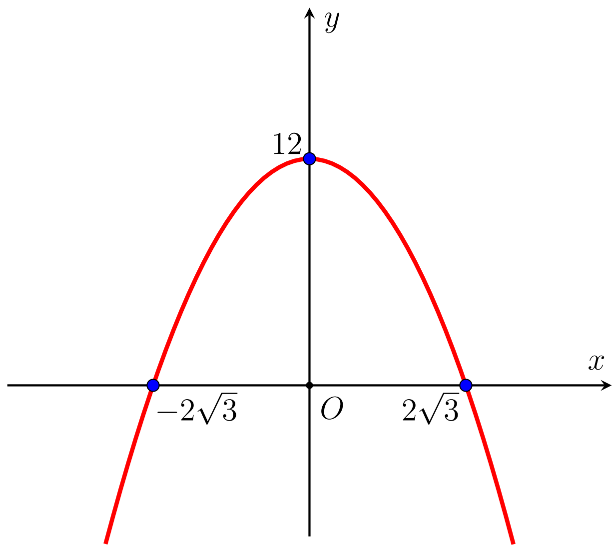
2–9) $x_1 = 3, \quad x_2 = \dfrac{1}{3}.$

2–10) $x_1 = 1 + \sqrt{2}, \quad x_2 = 1 - \sqrt{2}.$

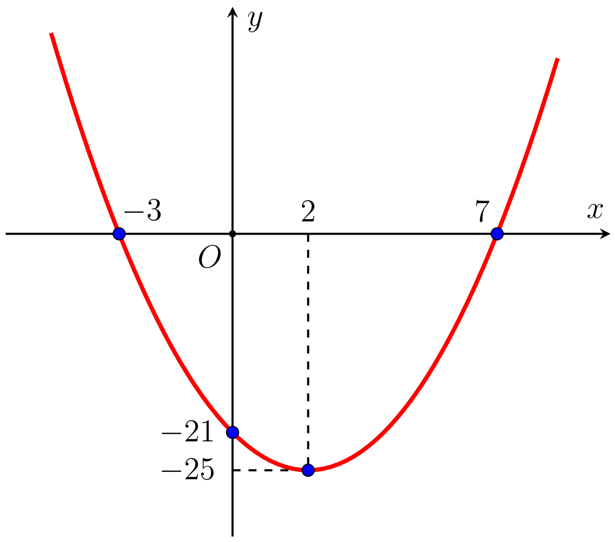
2–11)



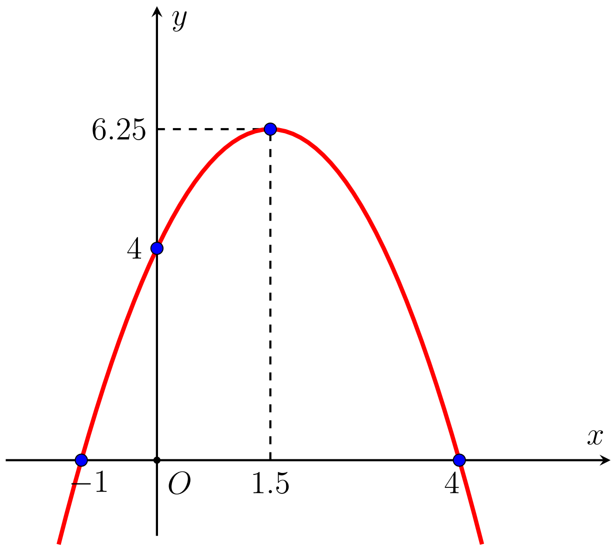
2–12)



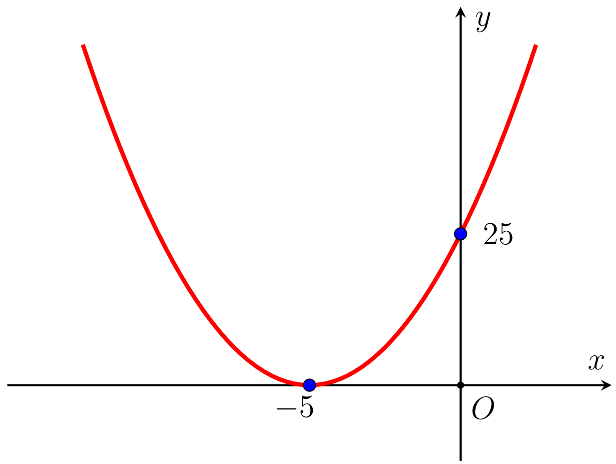
2–13)



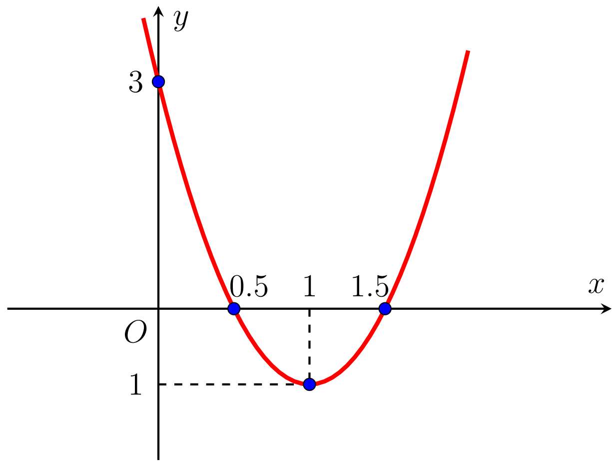
2–14)



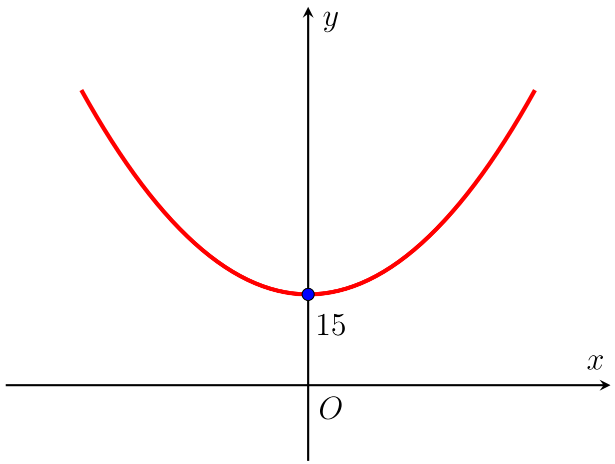
2–15)



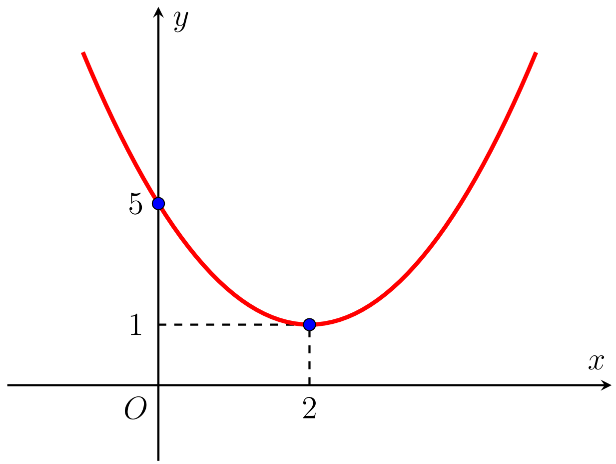
2–16)



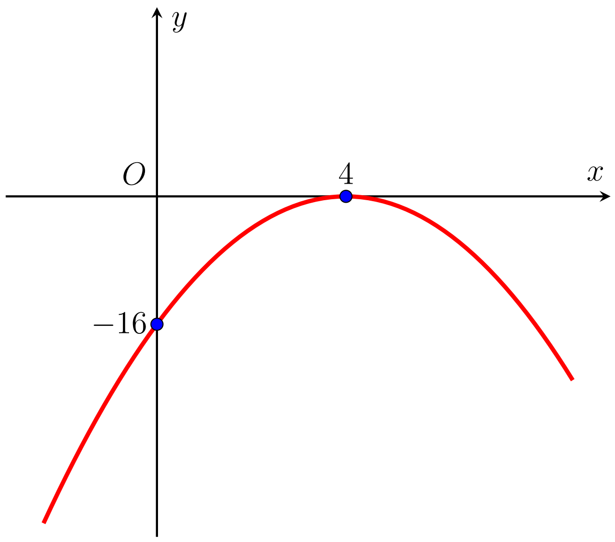
2–17)



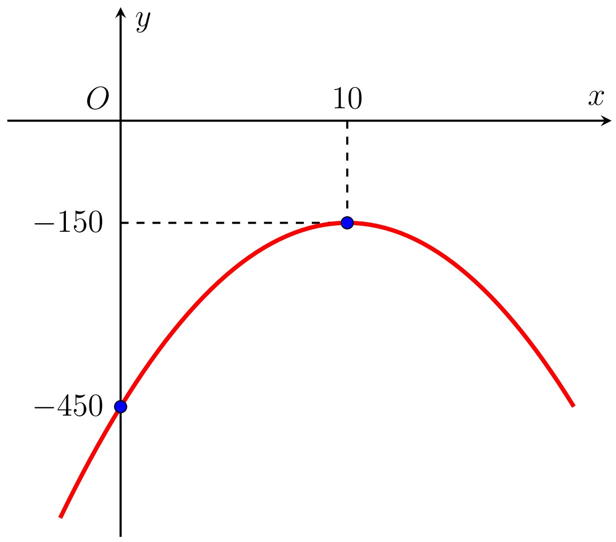
2–19)



2–18)



2–20)



Chapter 3

Exponential and Logarithmic Functions

Polynomials: A function of the form

$$p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a polynomial of degree n. For example, $120x^5 - 17x + \dfrac{7}{2}$ is a polynomial.

$\sqrt{x}, \quad x^{-1}, \quad \dfrac{1}{1+x}, \quad x^{5/3}$ are NOT polynomials.

Rational Functions: The quotient of two polynomials is a rational function $f(x) = \dfrac{p(x)}{q(x)}$. For example,

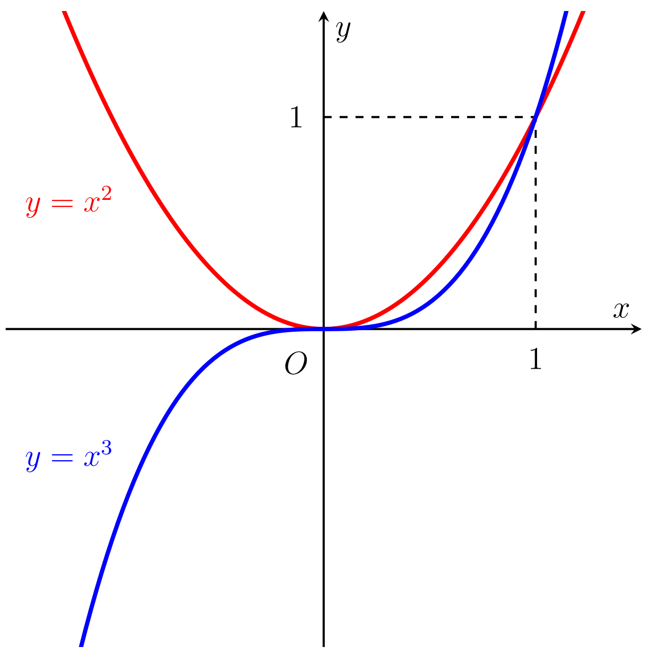
$$\frac{3x^2 - 5}{1 + 2x - 7x^3}$$

is a rational function.

Question: What is the domain of a polynomial? A rational function?

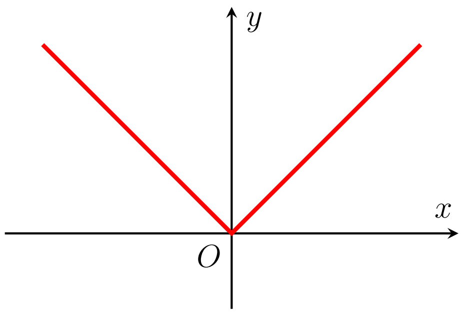
Example 3–1: Sketch the functions $y = x^2$ and $y = x^3$ on the same coordinate system.

Solution:

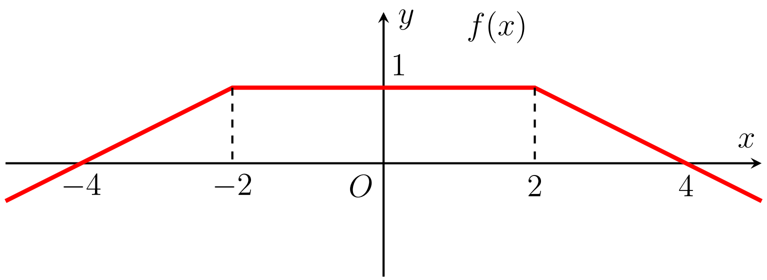


Piecewise-Defined Functions: We may define a function using different formulas for different parts of the domain. For example, the absolute value function is:

$$|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$$



Example 3–2: Find the formula of the function $f(x)$:



Solution:

$$f(x) = \begin{cases} \dfrac{x+4}{2} & \text{if} \quad x < -2 \\ 1 & \text{if} \quad -2 \leqslant x \leqslant 2 \\ \dfrac{-x+4}{2} & \text{if} \quad x > 2 \end{cases}$$

Inverse Functions:
If $f\big(g(x)\big) = x$ and $g\big(f(x)\big) = x$, the functions f and g are inverses of each other.

For example, the inverse of $f(x) = 2x + 1$ is:

$$f^{-1}(x) = g(x) = \frac{x-1}{2}$$

One–to–one Functions: If $f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$ then f is one-to one.

For example, $f(x) = x^3$ is one-to one but $g(x) = x^2$ is not, because $g(1) = g(-1)$.

Onto Functions: Let $f : A \to B$. If there exists an $x \in A$ for all $y \in B$ such that $f(x) = y$ then f is onto.

For example, $f(x) = 2x + 1$ is onto but $g(x) = |x|$ is not, because there is no x such that $g(x) = -2$ or any other negative number.

Theorem: A function has an inverse if and only if it is one-to-one and onto.

Example 3–3: Find the inverse of the function $f(x) = \dfrac{x-2}{x+1}$ on the domain $\mathbb{R} \setminus \{-1\}$ and range $\mathbb{R} \setminus \{1\}$.

Solution: $y = \dfrac{x-2}{x+1} \quad \Rightarrow \quad yx + y = x - 2$

$$yx - x = -y - 2 \quad \Rightarrow \quad x(y-1) = -y - 2$$

$$x = -\frac{y+2}{y-1}$$

In other words, $f^{-1}(x) = -\dfrac{x+2}{x-1}$.

Exponential Functions: Functions of the form

$$f(x) = a^x$$

where a is a positive constant (but $a \neq 1$) are called exponential functions. The domain is:

$$\mathbb{R} = (-\infty, \infty)$$

and the range is

$$(0, \infty)$$

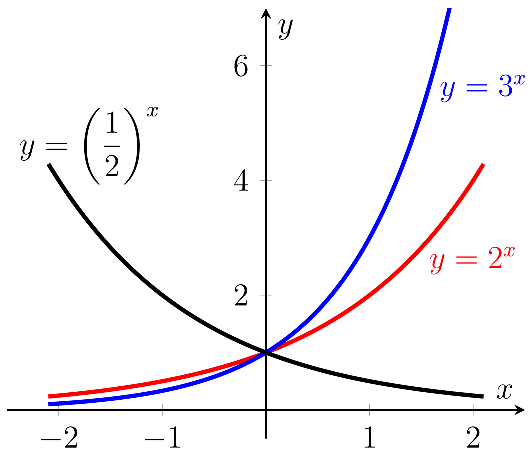
Remember that:

- $a^n = a \cdot a \cdots a$
- $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$
- $a^{1/n} = \sqrt[n]{a}$
- $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

The natural exponential function is:

$$f(x) = e^x$$

where $e = 2.71828\ldots$.



For the exponential function $f(x) = 2^x$,

$$\begin{aligned} f(6) &= 64 \\ f(5) &= 32 \\ f(1) &= 2 \\ f(0) &= 1 \\ f\left(\frac{1}{2}\right) &= \sqrt{2} \end{aligned}$$

Do not confuse this with the polynomial function $g(x) = x^2$ because

$$\begin{aligned} g(6) &= 36 \\ g(5) &= 25 \\ g(1) &= 1 \\ g(0) &= 0 \\ g\left(\frac{1}{2}\right) &= \frac{1}{4} \end{aligned}$$

Example 3–4: If we invest an amount A in the bank, and if the rate of interest is 15% per year, how much money will we have after n years?

Solution: We are multiplying by 1.15 every year, so: $1.15^n A$.

Example 3–5: A firm has C customers now. Every month, 30% of the customers leave. How many remain after n months?

Solution: We are multiplying by 0.7 every month, so: $0.7^n C$.

Logarithmic Functions: The inverse of the exponential function $y = a^x$ is the logarithmic function with base a:

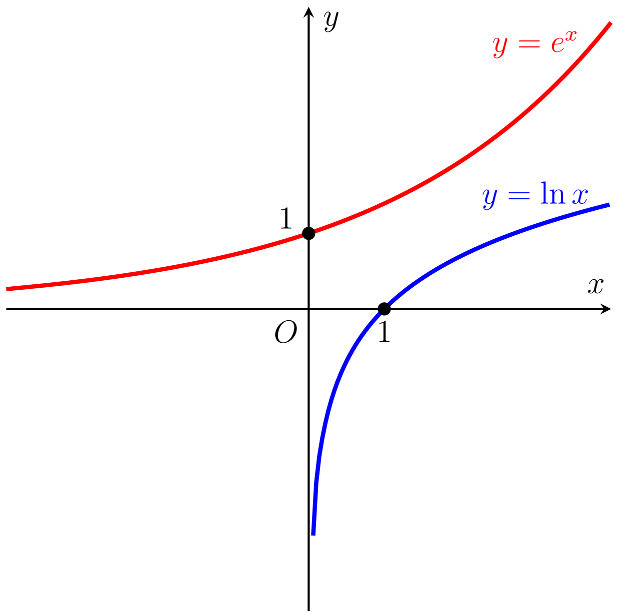
$$y = \log_a x$$

where $a > 0, \quad a \neq 1.$

$$a^{\log_a x} = \log_a(a^x) = x$$

We will use:

- $\log x$ for $\log_{10} x$ (common logarithm)
- $\ln x$ for $\log_e x$ (natural logarithm)



We can easily see that,

$$a^x \cdot a^y = a^{x+y} \quad \Rightarrow \quad \log_a(AB) = \log_a A + \log_a B$$

As a result of this,

- $\log_a \left(\dfrac{A}{B}\right) = \log_a A - \log_a B$
- $\log_a \left(\dfrac{1}{B}\right) = -\log_a B$
- $\log_a (A^r) = r \log_a A$

Any logarithm can be expressed in terms of the natural logarithm:

$$\log_a(x) = \frac{\ln x}{\ln a}$$

Any exponential can be expressed in terms of the natural exponential:

$$a^x = e^{x \ln a}$$

Example 3–6: Simplify $\log 360$.

Solution: First, we have to find factors of 360:

$360 = 2^3 \cdot 3^2 \cdot 5$

Now, we can use the properties of logarithms:

$$\begin{aligned} \log 360 &= \log 2^3 + \log 3^2 + \log 5 \\ &= 3\log 2 + 2\log 3 + 1 - \log 2 \\ &= 1 + 2\log 2 + 2\log 3 \end{aligned}$$

EXERCISES

Sketch the graphs of the following piecewise-defined functions:

3–1) $f(x) = \begin{cases} 2x & \text{if} \quad x < 5 \\ 10 & \text{if} \quad x \geqslant 5 \end{cases}$

3–2) $f(x) = \begin{cases} x+3 & \text{if} \quad x < 4 \\ x-1 & \text{if} \quad x \geqslant 4 \end{cases}$

Are the following functions polynomials?

3–3) $f(x) = 8x^4 + 1$

3–4) $f(x) = \dfrac{1-x}{x}$

3–5) $f(x) = \dfrac{1}{5}x + \dfrac{1}{3}$

3–6) $f(x) = 5x^5 - 3x^{2/3}$

Are the following functions one-to-one? Are they onto?

3–7) $f(x) = 2x$

3–8) $f(x) = x^3$

3–9) $f(x) = x^4 + x^2 + 1$

3–10) $f(x) = e^{2x}$

Find the inverse of the following functions.

3–11) $f(x) = 3x - 2$

3–12) $f(x) = \dfrac{x+2}{5x+4}$

3–13) $f(x) = \dfrac{1}{x}$

3–14) $f(x) = x^3 + 1$

Simplify the following:

3–15) $\log 400$

3–16) $\log 288$

3–17) $\log_9 27$

3–18) $\log_8 16$

3–19) $\log_2 1250$

3–20) $\log_3 \dfrac{\sqrt{3}}{81}$

3–21) $e^{2x+5\ln x}$

3–22) $\ln \dfrac{e}{\sqrt[3]{e}}$

3–23) $2^{3x+4\log_2 x}$

3–24) $3^{2\log_9 x}$

3–25) $5^{\log_{25} x}$

3–26) $10^{1+\log(2x)}$

Solve the following equations.

3–27) $5 = \left(5\sqrt{5}\right)^x$

3–28) $\log_x 12 = \dfrac{1}{2}$

3–29) $\log_x 77 = -1$

3–30) $\log_x 2 = 3$

3–31) $\log_x 64 = 4$

3–32) $\log_3 x = 5$

3–33) $\log_9(18x) = 2$

3–34) $\log_5 x = -\dfrac{1}{2}$

3–35) $\log(\log x) = 0$

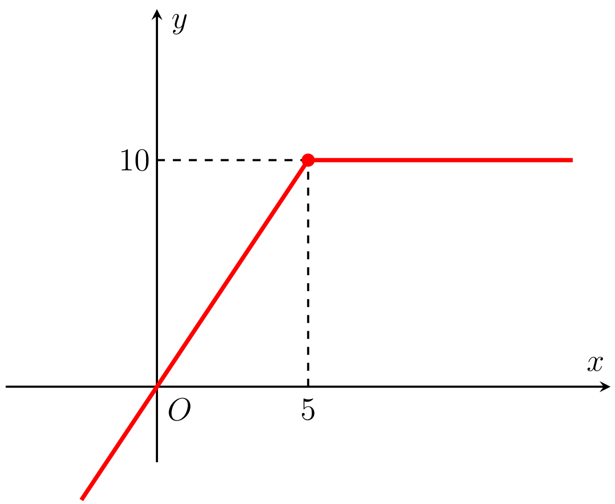
3–36) $\ln(\ln x) = 1$

3–37) $2^x = 100$

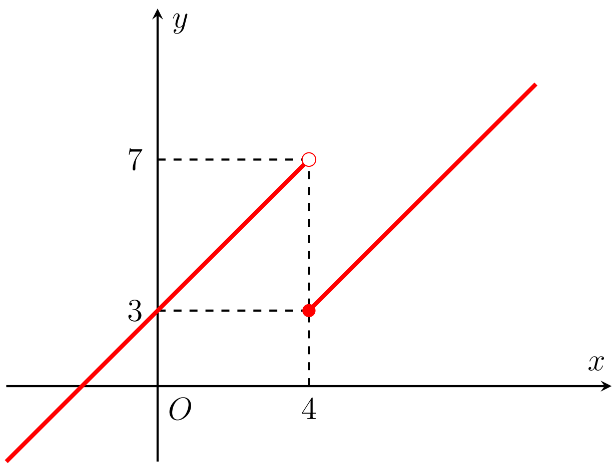
3–38) $2^{4x+4} = 8^{x-1}$

ANSWERS

3–1)



3–2)



3–3) Yes

3–4) No

3–5) Yes

3–6) No

3–7) One-to-one and onto.

3–8) One-to-one and onto.

3–9) Not one-to-one and not onto.

3–10) One-to-one and not onto.

3–11) $f^{-1}(x) = \dfrac{x+2}{3}$

3–12) $f^{-1}(x) = \dfrac{4x-2}{1-5x}$

3–13) $f^{-1}(x) = \dfrac{1}{x}$

3–14) $f^{-1}(x) = \sqrt[3]{x-1}$

3–15) $2 + 2\log 2$

3–16) $2\log 3 + 5\log 2$

3–17) $\frac{3}{2}$

3–18) $\frac{4}{3}$

3–19) $1 + 4\log_2 5$

3–20) $-\frac{7}{2}$

3–21) $x^5 e^{2x}$

3–22) $\frac{2}{3}$

3–23) $x^4 8^x$

3–24) x

3–25) $\sqrt{x}$

3–26) $20x$

3–27) $x = \frac{2}{3}$

3–28) $x = 144$

3–29) $x = \frac{1}{77}$

3–30) $x = 2^{1/3}$

3–31) $x = 2\sqrt{2}$

3–32) $x = 243$

3–33) $x = \frac{9}{2}$

3–34) $x = \frac{1}{\sqrt{5}}$

3–35) $x = 10$

3–36) $x = e^e$

3–37) $x = \frac{2}{\log 2}$

3–38) $x = -7$

Chapter 4

Limits

We say that $f(x)$ has the limit L at $x = a$ if $f(x)$ gets as close to L as we like, when x approaches a. (without getting equal to a)
We write this as:

$$\lim_{x \to a} f(x) = L$$

Example 4–1: Investigate the limit

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Solution:

x	f
0.9	1.9
0.99	1.99
0.999	1.999
$\vdots$	$\vdots$

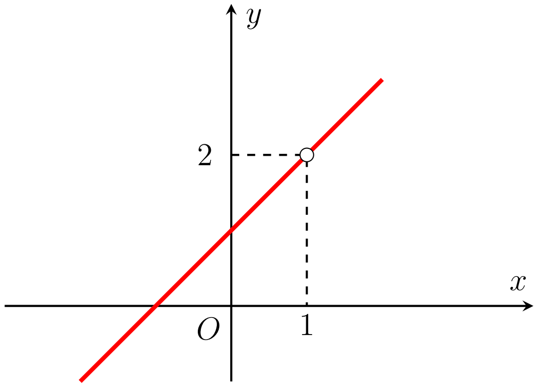
x	f
1.1	2.1
1.01	2.01
1.001	2.001
$\vdots$	$\vdots$

These results suggest that the limit is 2.

Actually, the function can be written as:

$$f(x) = \begin{cases} x + 1 & \text{if} \quad x \neq 1 \\ \text{undefined} & \text{if} \quad x = 1 \end{cases}$$

Its graph is:



Limit Laws: If both of the limits

$$\lim_{x\to a} f(x) = L \quad \text{and}$$

$$\lim_{x\to a} g(x) = M$$

exist, then:

- $\lim_{x\to a} f \pm g = L \pm M$
- $\lim_{x\to a} fg = LM$
- $\lim_{x\to a} \frac{f}{g} = \frac{L}{M}$ (if $M \neq 0$)
- $\lim_{x\to a} \sqrt[n]{f} = \sqrt[n]{L}$
- $\lim_{x\to a} f\big(g(x)\big) = f(M)$
  (If f is continuous at M)

Example 4–2: Evaluate the limit $\lim_{x\to 2} \dfrac{3}{x-2}$ (if it exists):

Solution: As x approaches 2, the function $\frac{3}{x-2}$ gets larger and larger without any bound. Therefore the limit does not exist. (Limit DNE.)

Example 4–3: Evaluate the following limit (if it exists):

$$\lim_{x\to -5} \frac{x^2+6x+5}{x^2+7x+10}$$

Solution:
$$\begin{aligned}\lim_{x\to -5} \frac{x^2+6x+5}{x^2+7x+10} &= \lim_{x\to -5} \frac{(x+5)(x+1)}{(x+5)(x+2)} \\ &= \lim_{x\to -5} \frac{(x+1)}{(x+2)} \\ &= \frac{4}{3}\end{aligned}$$

Example 4–4: Evaluate the following limit (if it exists):

$$\lim_{x\to 0} \frac{x^3-64}{x-4}$$

Solution: $\lim_{x\to 0} x^3 - 64 = -64$

$$\lim_{x\to 0} x - 4 = -4$$

Using limit laws, we obtain:

$$\lim_{x\to 0} \frac{x^3-64}{x-4} = \frac{-64}{-4} = 16$$

Example 4–5: Evaluate the limit $\lim_{x\to 4} \dfrac{x^3-64}{x-4}$ if it exists.

Solution: This question is different.

Although the limit $\lim_{x\to 4} x-4$ exists, it is zero, so we can NOT divide limit of the numerator by the limit of the denominator.

We have to use factorization:

$x^3-64=(x-4)(x^2+4x+16)$

$$\begin{aligned}
\lim_{x\to 4} \frac{x^3-64}{x-4} &= \lim_{x\to 4} \frac{(x-4)(x^2+4x+16)}{x-4} \\
&= \lim_{x\to 4} (x^2+4x+16) \\
&= 16+16+16 \\
&= 48
\end{aligned}$$

Example 4–6: Evaluate the limit (if it exists):

$$\lim_{x\to 5} \frac{1}{|x-5|}$$

Solution: $\dfrac{1}{|x-5|}$ increases without bounds as $x \to 5$.

Therefore limit does not exist.

Example 4–7: Evaluate the limit (if it exists):

$$\lim_{x\to -3} \frac{x^2+4x+3}{x^2+5x+6}$$

Solution:
$$\begin{aligned}
\lim_{x\to -3} \frac{x^2+4x+3}{x^2+5x+6} &= \lim_{x\to -3} \frac{(x+3)(x+1)}{(x+3)(x+2)} \\
&= \lim_{x\to -3} \frac{(x+1)}{(x+2)} \\
&= 2
\end{aligned}$$

Example 4–8: Evaluate the limit (if it exists)

$$\lim_{x\to 49} \frac{\sqrt{x}-7}{x-49}$$

Solution:
$$\begin{aligned}
\lim_{x\to 49} \frac{\sqrt{x}-7}{x-49} &= \lim_{x\to 49} \frac{\sqrt{x}-7}{x-49} \cdot \frac{\sqrt{x}+7}{\sqrt{x}+7} \\
&= \lim_{x\to 49} \frac{x-49}{(x-49)\left(\sqrt{x}+7\right)} \\
&= \lim_{x\to 49} \frac{1}{\sqrt{x}+7} \\
&= \frac{1}{14}
\end{aligned}$$

Example 4–9: Evaluate the limit (if it exists):

$$\lim_{x\to 2}\frac{x^3-7x+6}{x^2-5x+6}$$

Solution: This is of the form $\dfrac{0}{0}$, so, both the numerator and the denominator contain $(x-2)$.

Using polynomial division, we obtain:

$$\lim_{x\to 2}\frac{(x-2)(x^2+2x-3)}{(x-2)(x-3)}$$

For $x\neq 2$, this is:

$$=\lim_{x\to 2}\frac{(x^2+2x-3)}{(x-3)}=\frac{5}{-1}=-5$$

Example 4–10: Evaluate the limit (if it exists)

$$\lim_{x\to 8}\frac{2-\sqrt[3]{x}}{8-x}$$

Solution: Using the substitution $u=\sqrt[3]{x}$ we obtain:

$$\begin{aligned}\lim_{x\to 8}\frac{2-\sqrt[3]{x}}{8-x} &= \lim_{u\to 2}\frac{2-u}{8-u^3}\\ &= \lim_{u\to 2}\frac{2-u}{(2-u)(4+2u+u^2)}\\ &= \lim_{u\to 2}\frac{1}{4+2u+u^2}\\ &= \frac{1}{12}\end{aligned}$$

Example 4–11: Evaluate the limit (if it exists):

$$\lim_{x\to 3}\frac{x^2-9}{x^2-6x+9}$$

Solution: $=\lim_{x\to 3}\dfrac{(x-3)(x+3)}{(x-3)^2}=\lim_{x\to 3}\dfrac{(x+3)}{(x-3)}$

Limit does not exist.

Example 4–12: Evaluate the following limit (if it exists):

$$\lim_{x\to 0}\frac{\sqrt{9+12x}-3}{x}$$

Solution: Multiply both numerator and denominator by the conjugate of the numerator:

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{9+12x}-3}{x} &= \lim_{x\to 0}\frac{\sqrt{9+12x}-3}{x}\cdot\frac{\sqrt{9+12x}+3}{\sqrt{9+12x}+3}\\ &= \lim_{x\to 0}\frac{9+12x-9}{x\left(\sqrt{9+12x}+3\right)}\\ &= \lim_{x\to 0}\frac{12}{\sqrt{9+12x}+3}\\ &= 2\end{aligned}$$

EXERCISES

Evaluate the following limits (if they exist):

4–1) $\lim\limits_{x\to 2} \dfrac{x^3-8}{x-2}$

4–2) $\lim\limits_{x\to -4} \dfrac{x^2+11x+28}{x^2+12x+32}$

4–3) $\lim\limits_{x\to 5} \dfrac{x^2-25}{x^2-10x+25}$

4–4) $\lim\limits_{x\to 1} \dfrac{x^3-x^2-x+1}{x^2-x}$

4–5) $\lim\limits_{x\to 3} \dfrac{x^4-81}{x^3-27}$

4–6) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+16}-4}{x}$

4–7) $\lim\limits_{x\to 3} \dfrac{x-3}{\sqrt{x+6}-3}$

4–8) $\lim\limits_{x\to 64} \dfrac{\sqrt{x}-8}{x-64}$

4–9) $\lim\limits_{x\to 0} \dfrac{\sqrt{2x+1}-3}{x}$

4–10) $\lim\limits_{x\to 7} \dfrac{\sqrt{4x+8}-6}{x-7}$

Evaluate the following limits (if they exist):

4–11) $\lim\limits_{x\to \infty} \dfrac{x(x^2-5x+14)}{7-4x^3}$

4–12) $\lim\limits_{x\to \infty} \dfrac{3x^2+12x+9)}{(x^2-1)(x^2+1)}$

4–13) $\lim\limits_{x\to \infty} \dfrac{2+3x-4x^4}{\sqrt{x}\left(1-17x+8x^3\right)}$

4–14) $\lim\limits_{x\to \infty} \sqrt{x^2+6x}-x$

4–15) $\lim\limits_{x\to \infty} \sqrt{x^2+4x}-\sqrt{x^2-10x+1}$

4–16) $\lim\limits_{x\to \infty} \dfrac{x^4-16}{(2x-1)(2x+1)(x^2+1)}$

4–17) $\lim\limits_{x\to \infty} \sqrt{x^2-12x+24}-\sqrt{x^2+10x+5}$

4–18) $\lim\limits_{x\to \infty} \dfrac{1}{2x-\sqrt{4x^2-5x+6}}$

4–19) $\lim\limits_{x\to 3} \dfrac{x^2-9}{x^2-7x+12}$

4–20) $\lim\limits_{x\to 3} \dfrac{x^2-9}{\left(x-3\right)^2}$

Evaluate the following limits (if they exist):

4–21) $\displaystyle\lim_{x\to 3} \frac{(x-3)^2}{x^2-9}$

4–22) $\displaystyle\lim_{x\to 1} \frac{x^2-9}{x-3}$

4–23) $\displaystyle\lim_{x\to 0} \frac{|x|}{x}$

4–24) $\displaystyle\lim_{x\to 1} \frac{1}{x^2-1}$

4–25) $\displaystyle\lim_{x\to 1} \frac{x^3-1}{x^4-1}$

4–26) $\displaystyle\lim_{x\to 4} \frac{2-\sqrt{x}}{4-x}$

4–27) $\displaystyle\lim_{x\to 0} \frac{x^4-5x^2+12x+7}{5x^2+6}$

4–28) $\displaystyle\lim_{x\to 2} \frac{x^2-7x+10}{x^2-5x+6}$

4–29) $\displaystyle\lim_{x\to 4} \frac{x^2-7x+10}{x^2-5x+6}$

4–30) $\displaystyle\lim_{x\to 6} \frac{x^2-5x+4}{x-6}$

Evaluate the following limits (if they exist):

4–31) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

4–32) $\displaystyle\lim_{x\to -2} \frac{(x+2)^2}{x^4-16}$

4–33) $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x^2-9}$

4–34) $\displaystyle\lim_{x\to 1} \frac{1-\sqrt{x}}{1-\sqrt[3]{x}}$

4–35) $\displaystyle\lim_{x\to c} \frac{x^4-c^4}{x^3-c^3}$

4–36) $\displaystyle\lim_{x\to 0} \frac{x}{\sqrt{a+bx}-\sqrt{a-cx}}$

4–37) $\displaystyle\lim_{x\to 1} \frac{x^n-1}{x-1}$

4–38) $\displaystyle\lim_{x\to \infty} \frac{1}{\ln(x^2)}$

4–39) $\displaystyle\lim_{x\to \infty} \frac{8e^x}{4+5e^x}$

4–40) $\displaystyle\lim_{x\to \infty} \sqrt{2x^2-1}-\sqrt{x^2+1}$

ANSWERS

4–1) 12

4–2) $\frac{3}{4}$

4–3) Limit DNE.
(Limit does not exist.)

4–4) 0

4–5) 4

4–6) $\frac{1}{8}$

4–7) 6

4–8) $\frac{1}{16}$

4–9) Limit DNE.

4–10) $\frac{1}{3}$

4–11) $-\frac{1}{4}$

4–12) 0

4–13) $-\infty$

4–14) 3

4–15) 7

4–16) $\frac{1}{4}$

4–17) -11

4–18) $\frac{4}{5}$

4–19) -6

4–20) Limit DNE.

4–21) 0

4–22) 4

4–23) Limit DNE.

4–24) Limit DNE.

4–25) $\frac{3}{4}$

4–26) $\frac{1}{4}$

4–27) $\frac{7}{6}$

4–28) 3

4–29) -1

4–30) Limit DNE.

4–31) 1

4–32) 0

4–33) $\frac{1}{24}$

4–34) $\frac{3}{2}$

4–35) $\frac{4}{3}c$

4–36) $\frac{2\sqrt{a}}{b+c}$

4–37) n

4–38) 0

4–39) $\frac{8}{5}$

4–40) ∞

Chapter 5

One Sided Limits, Continuity

One Sided Limits: If x approaches a from right, taking values larger than a only, we denote this by $x \to a^+$. If $f(x)$ approaches L as $x \to a^+$, then we say that L is the right-hand limit of f at a.

$$\lim_{x \to a^+} f(x) = L$$

We define the left-hand limit of f at a similarly:

$$\lim_{x \to a^-} f(x) = L$$

Theorem: The limit $\lim_{x \to a} f(x) = L$ exists if and only if both one sided limits

$$\lim_{x \to a^+} f(x) \quad \text{and} \quad \lim_{x \to a^-} f(x)$$

exist and are equal to L.

Example 5–1: Consider the function

$$f(x) = \begin{cases} 2x - 1 & \text{if} \quad x < 1 \\ 5x - 2 & \text{if} \quad x > 1 \end{cases}$$

Find the limits $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1} f(x)$.

Solution: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x - 1 = 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 5x - 2 = 3$

$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$ therefore $\lim_{x \to 1} f(x)$ does not exist.

Example 5–2: Find the limits $\lim_{x\to 3^+} f(x)$ and $\lim_{x\to 3^-} f(x)$ and graph the function:

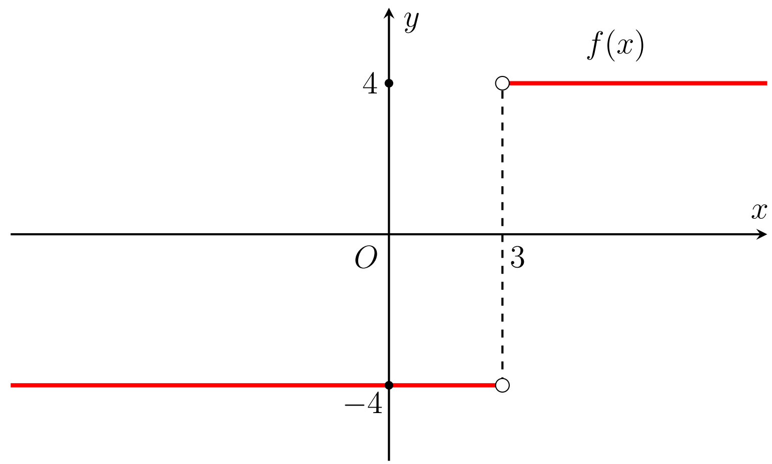
$$f(x) = \frac{4x-12}{|x-3|}$$

Solution: As $x \to 3^+$, $x-3>0$ therefore $|x-3| = x-3$ and

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{4x-12}{x-3} = 4$$

Similarly,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{4x-12}{-(x-3)} = -4$$



We can see that left and right limits exist at $x=3$.

But the limit $\lim_{x\to 3} f(x)$ does NOT exist.

Also, note that $f(3)$ is undefined.

Example 5–3: Let $f(x) = \begin{cases} 2-x^2 & \text{if} \quad x<0 \\ 7 & \text{if} \quad x=0 \\ e^x+e^{-x} & \text{if} \quad x>0 \end{cases}$

Find the limits $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0} f(x)$.

Solution: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2-x^2 = 2$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^x+e^{-x} = 2$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 2$ therefore $\lim_{x\to 0} f(x) = 2$.

(Note that the function value $f(0)=7$ does not have any effect on the limit.)

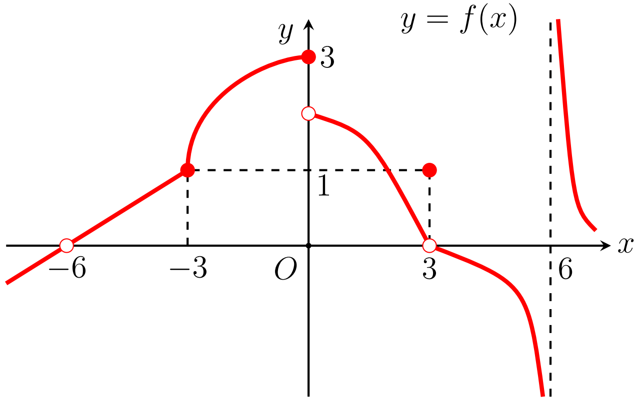
Example 5–4: Find the limit $\lim_{x\to 0^+} \ln x$ if it exists.

Solution: Checking the graph of $f(x) = \ln x$ we see that:

$$\lim_{x\to 0^+} \ln x = -\infty$$

Note that the question $\lim_{x\to 0} \ln x$ would be meaningless.

Example 5–5: Find the limits based on the function $f(x)$ in the figure: (If they exist.)



a) $\lim_{x\to -6^-} f(x)$, $\lim_{x\to -6^+} f(x)$, $\lim_{x\to -6} f(x)$.

b) $\lim_{x\to -3^-} f(x)$, $\lim_{x\to -3^+} f(x)$, $\lim_{x\to -3} f(x)$.

c) $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$.

d) $\lim_{x\to 3^-} f(x)$, $\lim_{x\to 3^+} f(x)$, $\lim_{x\to 3} f(x)$.

e) $\lim_{x\to 6^-} f(x)$, $\lim_{x\to 6^+} f(x)$, $\lim_{x\to 6} f(x)$.

Solution:

a) $0,\quad 0,\quad 0.$

b) $1,\quad 1,\quad 1.$

c) $3,\quad 2,\quad$ does not exist.

d) $0,\quad 0,\quad 0.$

e) $-\infty,\quad \infty,\quad$ does not exist.

Example 5–6: Evaluate the limit (if it exists)

$$\lim_{x\to 8^+} \frac{x^2-10x+16}{\sqrt{x-8}}$$

Solution: Note that square root of a negative number is not defined, so x should not take values less than 8.

Therefore the question

$$\lim_{x\to 8} \frac{x^2-10x+16}{\sqrt{x-8}}$$

would be meaningless.

Now if we factor $x^2-10x+16$ as:

$$\begin{aligned} x^2-10x+16 &= (x-8)(x-2) \\ &= \sqrt{x-8}\,\sqrt{x-8}\,(x-2) \end{aligned}$$

we obtain:

$$\begin{aligned} \lim_{x\to 8^+} \frac{x^2-10x+16}{\sqrt{x-8}} &= \lim_{x\to 8^+} \frac{\sqrt{x-8}\,\sqrt{x-8}\,(x-2)}{\sqrt{x-8}} \\ &= \lim_{x\to 8^+} \sqrt{x-8}\,(x-2) \\ &= 0 \end{aligned}$$

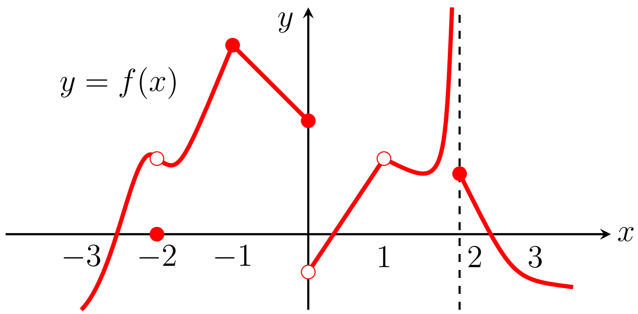
Continuity: We say that f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

In other words:

- f must be defined at a.
- $\lim\limits_{x \to a} f(x)$ must exist.
- The limit must be equal to the function value.

Example 5–7: Determine the points where $f(x)$ is discontinuous:



Solution: $f(x)$ is discontinuous at:

- $x = -2$, limit and function value are different.
- $x = 0$, limit does not exist.
- $x = 1$, function is undefined.
- $x = 2$, limit does not exist.

Example 5–8: Let $f(x) = \begin{cases} 2x^2 + a & \text{if} \quad x < 2 \\ b & \text{if} \quad x = 2 \\ 3x - 2 & \text{if} \quad x > 2 \end{cases}$

Find a and b if $f(x)$ is continuous at $x = 2$.

Solution: $\lim\limits_{x \to 2^-} f(x) = 8 + a$ and $\lim\limits_{x \to 2^+} f(x) = 4$.

If f is continuous at $x = 2$, then

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$\Rightarrow \quad 8 + a = b = 4$$

We find $a = -4, \quad b = 4$.

Example 5–9: Let $f(x) = 2 + 12x - x^3 + 20x^4$. Find the points where $f(x)$ is discontinuous.

Solution: The given function is a polynomial. A polynomial function is continuous at all points.

Example 5–10: Let $f(x) = \dfrac{3x - 2}{x^2 + 4}$. Find the points where $f(x)$ is discontinuous.

Solution: This is a rational function. A rational function is discontinuous only at the points where denominator is zero. But the equation

$$x^2 + 4 = 0$$

has no solutions. This means there is no discontinuity. In other words $f(x)$ is continuous on $\mathbb{R}$.

Example 5–11: Let $f(x) = \dfrac{x-2}{x^2-7x+10}$.

Find the points where $f(x)$ is discontinuous.

Solution: This is a rational function. So:

$$x^2 - 7x + 10 = 0 \quad \Rightarrow \quad x = 2, \ x = 5$$

$f(x)$ is discontinuous at $x = 2$ and $x = 5$.

Example 5–12: Let $f(x) = \begin{cases} \log\left(\dfrac{x}{2} + b\right) & \text{if} \quad x < 8 \\ x\left(\sqrt{x-8} + \dfrac{1}{4}\right) & \text{if} \quad x \geqslant 8 \end{cases}$

Find b if $f(x)$ is continuous at $x = 8$.

Solution: $\lim\limits_{x \to 8^+} f(x) = 2$

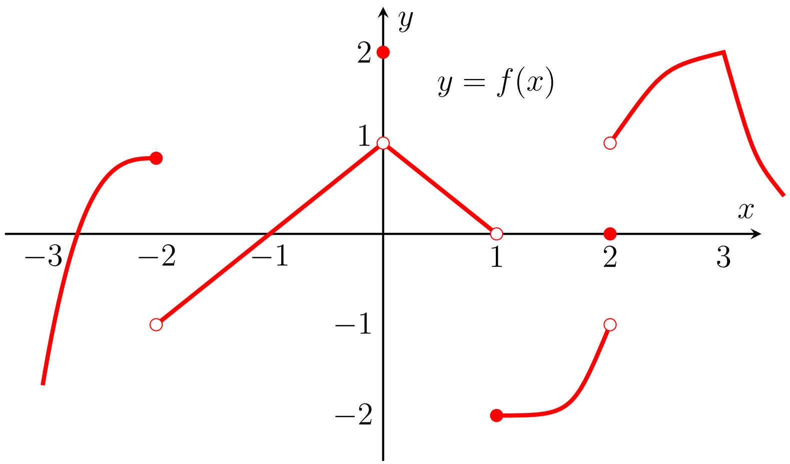
$\lim\limits_{x \to 8^-} f(x) = \log(4 + b)$

If f is continuous, these limits must be equal.

$$\begin{aligned} \log(4+b) &= 2 \\ 4 + b &= 100 \\ b &= 96 \end{aligned}$$

EXERCISES

5–1) Find the limits based on the figure:



a) $\lim\limits_{x \to -2^-} f(x)$, $\lim\limits_{x \to -2^+} f(x)$, $\lim\limits_{x \to -2} f(x)$.

b) $\lim\limits_{x \to -1^-} f(x)$, $\lim\limits_{x \to -1^+} f(x)$, $\lim\limits_{x \to -1} f(x)$.

c) $\lim\limits_{x \to 0^-} f(x)$, $\lim\limits_{x \to 0^+} f(x)$, $\lim\limits_{x \to 0} f(x)$.

d) $\lim\limits_{x \to 1^-} f(x)$, $\lim\limits_{x \to 1^+} f(x)$, $\lim\limits_{x \to 1} f(x)$.

e) $\lim\limits_{x \to 2^-} f(x)$, $\lim\limits_{x \to 2^+} f(x)$, $\lim\limits_{x \to 2} f(x)$.

f) $\lim\limits_{x \to 3^-} f(x)$, $\lim\limits_{x \to 3^+} f(x)$, $\lim\limits_{x \to 3} f(x)$.

5–2) Find the points where $f(x)$ of previous question is discontinuous.

Evaluate the following limits: (If they exist)

5–3) $\lim_{x\to 7^-} \dfrac{2}{x-7}$

5–4) $\lim_{x\to 7^+} \dfrac{2}{x-7}$

5–5) $\lim_{x\to 7^-} \dfrac{|x-7|}{x-7}$

5–6) $\lim_{x\to 7^+} \dfrac{|x-7|}{x-7}$

5–7) $\lim_{x\to 3^+} \sqrt{\dfrac{x-3}{x+3}}$

5–8) $\lim_{x\to 0^+} \dfrac{\sqrt{16+3x}-4}{x}$

5–9) $\lim_{x\to -2^+} \dfrac{|x^2-4|}{x+2}$

5–10) $\lim_{x\to -2^-} \dfrac{|x^2-4|}{x+2}$

5–11) $\lim_{x\to 0^+} \dfrac{2x^2+3x|x|}{x|x|}$

5–12) $\lim_{x\to 0^-} \dfrac{2x^2+3x|x|}{x|x|}$

Find all the discontinuities of the following functions:

5–13) $f(x) = \dfrac{x^2-2}{x^2-4}$

5–14) $f(x) = \dfrac{|x-a|}{x-a}$

5–15) $f(x) = \dfrac{x^2-5x+6}{x^2-4x+3}$

5–16) $f(x) = \dfrac{1}{e^{2x}-e^{3x}}$

5–17) $f(x) = \dfrac{x-5}{x^2-25}$

5–18) $f(x) = \dfrac{1}{1-|x|}$

5–19) $f(x) = \begin{cases} -1+x & \text{if} \quad x \leqslant 0 \\ 1+x^2 & \text{if} \quad x > 0 \end{cases}$

5–20) $f(x) = \begin{cases} 12x-20 & \text{if} \quad x < 2 \\ 8 & \text{if} \quad x = 2 \\ x^2 & \text{if} \quad x > 2 \end{cases}$

Find the values of constants that will make the following functions continuous everywhere:

5–21) $f(x) = \begin{cases} a + bx^2 & \text{if} \quad x < 0 \\ b & \text{if} \quad x = 0 \\ 2 + e^{-x} & \text{if} \quad x > 0 \end{cases}$

5–22) $f(x) = \begin{cases} cx^2 - 2 & \text{if} \quad x \leqslant 2 \\ \dfrac{x}{c} & \text{if} \quad x > 2 \end{cases}$

5–23) $f(x) = \begin{cases} x^2 - c^2 & \text{if} \quad x \leqslant 1 \\ (x-c)^2 & \text{if} \quad x > 1 \end{cases}$

5–24) $f(x) = \begin{cases} e^{ax} & \text{if} \quad x \leqslant 0 \\ \ln\left(b + x^2\right) & \text{if} \quad x > 0 \end{cases}$

ANSWERS

5–1)

a) 1, -1, Does Not Exist.

b) 0, 0, 0.

c) 1, 1, 1.

d) 0, -2, DNE.

e) -1, 1, DNE.

f) 2, 2, 2.

5–2)

$x = -2$.

$x = 0$.

$x = 1$.

$x = 2$.

5–3) $-\infty$

5–4) ∞

5–5) -1

5–6) 1

5–7) 0

5–8) $\dfrac{3}{8}$

5–9) 4

5–10) -4

5–11) 5

5–12) 1

5–13) $x = 2$ and $x = -2$.

5–14) $x = a$.

5–15) $x = 1$, $x = 3$.

5–16) $x = 0$.

5–17) $x = -5$, $x = 5$.

5–18) $x = 1$ and $x = -1$.

5–19) $x = 0$.

5–20) $x = 2$.

5–21) $a = b = 3$

5–22) $c = 1$, or $c = -\dfrac{1}{2}$

5–23) $c = 0$, or $c = 1$

5–24) $b = e$, a is arbitrary.

Chapter 6

Derivatives

Definition and Notation: The derivative of the function $f(x)$ is the function $f'(x)$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\text{Or, equivalently:} \quad f'(x) = \lim_{a \to x} \frac{f(x) - f(a)}{x - a}$$

We can think of the derivative as

- The rate of change of a function f, or
- The slope of the curve of $y = f(x)$.

We will use $y', \quad f'(x), \quad \dfrac{dy}{dx}, \quad \dfrac{d}{dx}f(x)$ to denote derivatives and $f'(a), \quad \left.\dfrac{dy}{dx}\right|_{x=a}$ to denote their values at a certain point.

Note that derivative is a function, its value at a point is a number.

Higher Order Derivatives: We can find the derivative of the derivative of a function. It is called second derivative and denoted by:

$$y'', \quad f''(x), \quad \frac{d^2y}{dx^2}.$$

For third derivative, we use $f'''(x)$ but for fourth and higher derivatives, we use the notation $f^{(4)}(x)$, $f^{(5)}(x)$ etc.

Example 6–1: Let $f(x) = 7x^3 - 18x$. Find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.

Solution:

$$\begin{aligned} f'(x) &= 21x^2 - 18 \\ f''(x) &= 42x \\ f'''(x) &= 42 \\ f^{(4)}(x) &= 0 \end{aligned}$$

Differentiation Formulas: Using the definition of derivative, we obtain:

- Derivative of a constant is zero, i.e.

$$\frac{dc}{dx} = 0$$

- Derivative of $f(x) = x$ is 1:

$$\frac{d}{dx} x = 1$$

- Derivative of $f(x) = x^2$ is $2x$:

$$\frac{d}{dx} x^2 = 2x$$

- Derivative of $f(x) = x^n$ is:

$$\frac{d}{dx} x^n = nx^{n-1}$$

- Derivative of $f(x) = \sqrt{x}$ is:

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$

- If f is a function and c is a constant, then

$$(cf)' = cf'$$

- If f and g are functions, then

$$(f+g)' = f' + g'$$

Example 6–2: Evaluate the derivative of $f(x) = \dfrac{7x^3 - 18x}{x}$.

Solution: First we have to simplify:

$f(x) = 7x^2 - 18$

Then we use the differentiation rules:

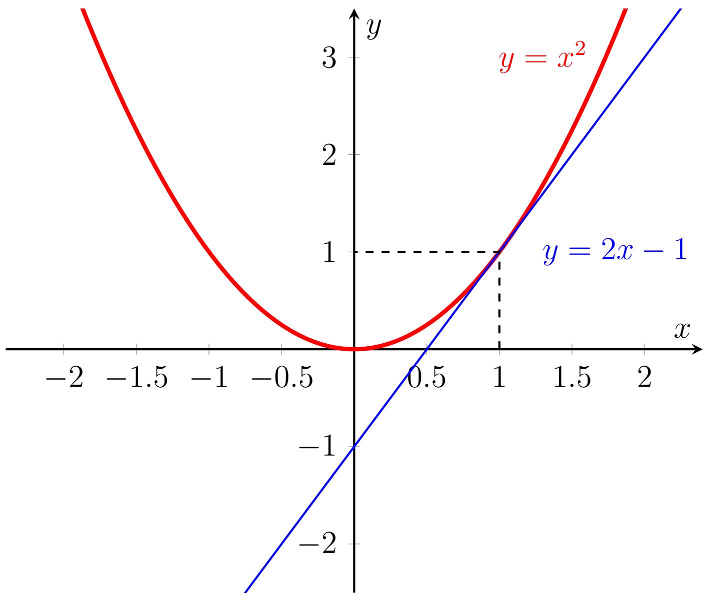
$f'(x) = 14x$

Example 6–3: Find the equation of the tangent line to the graph of $f(x) = x^2$ at the point $(1, 1)$.

Solution: $f'(x) = 2x \quad \Rightarrow \quad m = f'(1) = 2$

Using point slope equation $\Big(y - y_0 = m(x - x_0)\Big)$ we find the equation of the tangent line as:

$(y-1) = 2(x-1) \quad \Rightarrow \quad y = 2x - 1$



Differentiation Rules:

Product Rule: If f and g are differentiable at x, then fg is differentiable at x and

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

or more briefly:

$$(fg)' = f'g + fg'$$

Example 6–4: Find the derivative of $f(x) = (x^4+14x)(7x^3+17)$

Solution: $f'(x) = (4x^3+14)(7x^3+17) + (x^4+14x)\,21x^2$

Reciprocal Rule: If f is differentiable at x and if $f(x) \neq 0$ then:

$$\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}$$

Example 6–5: Using the reciprocal rule, find the derivative of $f(x) = \dfrac{1}{x^n}$.

Solution: $f'(x) = \dfrac{-nx^{n-1}}{x^{2n}} = -\dfrac{n}{x^{n+1}} = -n\,x^{-n-1}$

Example 6–6: Find the derivative of $f(x) = \dfrac{1}{8x^2+12x+1}$.

Solution: $f'(x) = -\dfrac{16x+12}{(8x^2+12x+1)^2}$

Quotient Rule: If f and g are differentiable at x, and $g(x) \neq 0$ then $\dfrac{f}{g}$ is differentiable at x:

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

Example 6–7: Find the derivative of $f(x) = \dfrac{2x+3}{5x^2+7}$.

Solution:
$$\begin{aligned} f'(x) &= \frac{2(5x^2+7) - 10x\,(2x+3)}{(5x^2+7)^2} \\ &= \frac{-10x^2-30x+14}{(5x^2+7)^2} \end{aligned}$$

Example 6–8: Find the derivative of $f(x) = \dfrac{1}{x^3+x}$.

Solution: The quotient rule gives:

$$\begin{aligned} f'(x) &= \frac{0\cdot(x^3+x) - (3x^2+1)\cdot 1}{(x^3+x)^2} \\ &= -\frac{3x^2+1}{(x^3+x)^2} \end{aligned}$$

Alternatively, we can use reciprocal rule:

$$\begin{aligned} f'(x) &= \frac{-(x^3+x)'}{(x^3+x)^2} \\ &= -\frac{3x^2+1}{(x^3+x)^2} \end{aligned}$$

Exponentials and Logarithms: The derivatives of exponential and logarithmic functions are

$$\frac{d}{dx}\, e^x = e^x \qquad \text{and} \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

e^x is the only nonzero function whose derivative is itself.

Example 6–9: Find the derivative of $f(x) = x^3 e^x$.

Solution: Using product rule,

$$f'(x) = 3x^2 e^x + x^3 e^x$$

Example 6–10: Find the derivative of $f(x) = e^x \ln x$.

Solution: Using product rule,

$$f'(x) = e^x \ln x + \frac{e^x}{x}$$

Example 6–11: Find the derivative of $f(x) = e^{-x}$.

Solution: We know that $e^{-x} = \dfrac{1}{e^x}$. Using reciprocal rule,

$$\begin{aligned} f'(x) &= \frac{-(e^x)'}{(e^x)^2} \\ &= -\frac{e^x}{e^{2x}} \\ &= -e^{-x} \end{aligned}$$

Example 6–12: Find the derivative of

$$f(x) = \frac{x^4}{e^x - x^2}$$

Solution: Using quotient rule,

$$\begin{aligned} f'(x) &= \frac{4x^3(e^x - x^2) - (e^x - 2x)x^4}{(e^x - x^2)^2} \\ &= \frac{4x^3e^x - 4x^5 - x^4e^x + 2x^5}{(e^x - x^2)^2} \\ &= \frac{(4x^3 - x^4)e^x - 2x^5}{(e^x - x^2)^2} \end{aligned}$$

Example 6–13: Find the derivative of

$$f(x) = \frac{1}{x - e^x + \ln x}$$

Solution: Using quotient rule,

$$\begin{aligned} f'(x) &= \frac{0 - \left(1 - e^x + \frac{1}{x}\right)}{(x - e^x + \ln x)^2} \\ &= -\frac{1 - e^x + \frac{1}{x}}{(x - e^x + \ln x)^2} \\ &= -\frac{x - xe^x + 1}{x(x - e^x + \ln x)^2} \end{aligned}$$

EXERCISES

Evaluate the derivatives of the following functions:

6–1) $f(x) = 1 - \sqrt{x}$

6–2) $f(x) = 4 + 3x - 12x^3$

6–3) $f(x) = x^{-1} + 4x^{-2}$

6–4) $f(x) = \dfrac{1}{\sqrt[4]{x}}$

6–5) $f(x) = \dfrac{1}{x} - \dfrac{2}{x^2}$

6–6) $f(x) = 20x^{-4} + 4x^{1/4}$

6–7) $f(x) = \dfrac{x^3 - x}{\sqrt{x}}$

6–8) $f(x) = \dfrac{2x^4 - x^3 - 1}{x}$

6–9) $f(x) = (x^2 + 2)(x^2 - 3)$

6–10) $f(x) = \dfrac{x}{x^2 + 4}$

6–11) $f(x) = \dfrac{x^2 + 12}{5x - 2}$

6–12) $f(x) = \dfrac{x^{3/2} + x^{-1/2}}{\sqrt{x} + \dfrac{1}{\sqrt{x}}}$

Evaluate the derivatives of the following functions:

6–13) $f(x) = x^{12}e^x$

6–14) $f(x) = x^2 \ln\left(x^3\right)$

6–15) $f(x) = \dfrac{5x}{\ln x}$

6–16) $f(x) = \dfrac{e^x}{1 + x^2}$

6–17) $f(x) = \dfrac{1}{1 + 2x + 3e^x}$

6–18) $f(x) = \dfrac{1}{e^x + 2\ln x}$

6–19) $f(x) = x^4 e^x \ln x$

6–20) $f(x) = \left(x + e^x\right)\left(x^2 + \ln x\right)$

6–21) $f(x) = \dfrac{4x^2 - 5x}{2e^x - 3x}$

6–22) $f(x) = \dfrac{1}{\ln\left(4x\right)}$

6–23) $f(x) = e^x e^x e^x$

6–24) $f(x) = \dfrac{2 - 3\ln x}{5\ln x + 1}$

ANSWERS

6–1) $f'(x) = \dfrac{-1}{2\sqrt{x}}$

6–2) $f'(x) = 3 - 36x^2$

6–3) $f'(x) = -x^{-2} - 8x^{-3}$

6–4) $f'(x) = -\dfrac{1}{4}x^{-5/4}$

6–5) $f'(x) = -\dfrac{1}{x^2} + \dfrac{4}{x^3}$

6–6) $f'(x) = -80x^{-5} + x^{-3/4}$

6–7) $f'(x) = \dfrac{5}{2}x^{3/2} - \dfrac{1}{2}x^{-1/2}$

6–8) $f'(x) = 6x^2 - 2x + \dfrac{1}{x^2}$

6–9) $f'(x) = 4x^3 - 2x$

6–10) $f'(x) = \dfrac{4 - x^2}{(x^2+4)^2}$

6–11) $f'(x) = \dfrac{5x^2 - 4x - 60}{(5x-2)^2}$

6–12) $f'(x) = \dfrac{x^2 + 2x - 1}{(x+1)^2}$

6–13) $f'(x) = 12x^{11}e^x + x^12e^x$

6–14) $f'(x) = 6x\ln x + 3x$

6–15) $f'(x) = \dfrac{5\ln x - 5}{\ln^2 x}$

6–16) $f'(x) = \dfrac{e^x(1 + x^2 - 2x)}{(1+x^2)^2}$

6–17) $f'(x) = -\dfrac{2 + 3e^x}{(1 + 2x + 3e^x)^2}$

6–18) $f'(x) = -\dfrac{e^x + \frac{2}{x}}{(e^x + 2\ln x)^2}$

6–19) $f'(x) = x^3e^x\big(4\ln x + x\ln x + 1\big)$

6–20) $f'(x) = \big(1 + e^x\big)\big(x^2 + \ln x\big) + \big(x + e^x\big)\left(2x + \dfrac{1}{x}\right)$

6–21) $f'(x) = \dfrac{(8x-5)(2e^x - 3x) - (2e^x - 3)(4x^2 - 5x)}{(2e^x - 3x)^2}$

6–22) $f'(x) = -\dfrac{1}{x\ln^2(4x)}$

6–23) $f'(x) = 3e^{3x}$

6–24) $f'(x) = -\dfrac{13}{\big(5\ln x + 1\big)^2}$

Chapter 7

Chain Rule

Chain Rule: If f and g are differentiable then $f(g(x))$ is also differentiable and

$$\Big[f\big(g(x)\big)\Big]' = f'\big(g(x)\big) \cdot g'(x)$$

or more briefly :

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example 7–1: Find $\dfrac{d}{dx}\big(3x^2+1\big)^5$.

Solution: Here $u = 3x^2+1$ and $y = u^5$. Using the above formula, we obtain:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 5u^4 \cdot 6x \\ &= 5(3x^2+1)^4 \cdot 6x \\ &= 30x(3x^2+1)^4\end{aligned}$$

Example 7–2: Find $f'(x)$ where $f(x) = e^{x^5}$.

Solution: Here $u = x^5$ and $f = e^u$. Using the chain rule formula, we obtain:

$$f'(x) = e^{x^5} \cdot 5x^4$$

Example 7–3: Find $f'(x)$ where $f(x) = \ln(1+2x+5x^2)$.

Solution: Here $u = 1+2x+5x^2$ and $f = \ln u$. Using chain rule:

$$\begin{aligned}f'(x) = \frac{df}{dx} &= \frac{df}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{u} \cdot (2+10x) \\ &= \frac{1}{1+2x+5x^2} \cdot (2+10x) \\ &= \frac{2+10x}{1+2x+5x^2}\end{aligned}$$

Example 7–4: Find the derivatives of the following functions using chain rule:

a) $f(x) = \sqrt{2x-3}$

b) $f(x) = \left(x^3 + e^x\right)^7$

c) $f(x) = \ln\left(\dfrac{x+1}{2x+1}\right)$

Solution:

a) Choose $u = 2x - 3 \quad \Rightarrow \quad \dfrac{du}{dx} = 2$

$$\begin{aligned} f'(x) &= \frac{1}{2}(2x-3)^{-1/2} \cdot 2 \\ &= \frac{1}{\sqrt{2x-3}} \end{aligned}$$

b) Choose $u = x^3 + e^x \quad \Rightarrow \quad \dfrac{du}{dx} = 3x^2 + e^x$

$$f'(x) = 7\left(x^3 + e^x\right)^6 \cdot (3x^2 + e^x)$$

c) Choose $u = \dfrac{x+1}{2x+1}$

$$\Rightarrow \quad \frac{du}{dx} = \frac{2x+1-2(x+1)}{(2x+1)^2} = -\frac{1}{(2x+1)^2}$$

$$\begin{aligned} f'(x) &= \frac{1}{\left(\dfrac{x+1}{2x+1}\right)} \cdot \frac{-1}{(2x+1)^2} \\ &= -\frac{1}{(x+1)(2x+1)} \end{aligned}$$

Example 7–5: Find the derivatives of the following functions using chain rule:

a) $f(x) = e^{ax}$

b) $f(x) = \ln\left(ax\right)$

c) $f(x) = e^{x^2 - x}$

d) $f(x) = \ln\left(x^8\right)$

Solution:

a) $u = ax \quad \Rightarrow \quad \dfrac{du}{dx} = a$

$f'(x) = ae^{ax}$

b) $u = ax \quad \Rightarrow \quad \dfrac{du}{dx} = a$

$f'(x) = \dfrac{1}{ax} \cdot a = \dfrac{1}{x}$

c) $u = x^2 - x \quad \Rightarrow \quad \dfrac{du}{dx} = 2x - 1$

$f'(x) = (2x-1)e^{x^2-x}$

d) $u = x^8 \quad \Rightarrow \quad \dfrac{du}{dx} = 8x^7$

$f'(x) = \dfrac{1}{x^8} \cdot 8x^7 = \dfrac{8}{x}$

Logarithmic Differentiation: Logarithm transforms products into sums. This helps in finding derivatives of some complicated functions.

For example if

$$y = \frac{(x^3+1)(x^2-1)}{x^8+6x^4+1}$$

then

$$\ln y = \ln(x^3+1) + \ln(x^2-1) - \ln(x^8+6x^4+1)$$

Derivative of both sides gives:

$$\frac{y'}{y} = \frac{3x^2}{x^3+1} + \frac{2x}{x^2-1} - \frac{8x^7+24x^3}{x^8+6x^4+1}$$

Example 7–6: Find the derivative of the function

$$y = f(x) = x^x$$

Solution: We can not use the power rule or product rule here. We have to use logarithms.

$$\begin{aligned} \ln y &= x \ln x \\ (\ln y)' &= \ln x + x \cdot \frac{1}{x} \\ \frac{y'}{y} &= \ln x + 1 \\ y' &= (\ln x + 1)\, x^x \end{aligned}$$

Example 7–7: Find the derivative of the function:

$$y = x^{\ln x}$$

Solution:

$$\begin{aligned} \ln y &= \ln x \cdot \ln x = \ln^2 x \\ (\ln y)' &= 2 \ln x \cdot \frac{1}{x} \\ \frac{y'}{y} &= \frac{2 \ln x}{x} \\ y' &= \frac{2 \ln x}{x} \cdot x^{\ln x} = 2x^{\ln x - 1} \ln x \end{aligned}$$

Example 7–8: Find the derivative of the function:

$$y = (x + e^x)^{\ln x}$$

Solution:

$$\begin{aligned} \ln y &= \ln x \ln(x + e^x) \\ (\ln y)' &= \frac{1}{x} \ln(x+e^x) + \frac{1+e^x}{x+e^x} \ln x \\ \frac{y'}{y} &= \frac{\ln(x+e^x)}{x} + \frac{1+e^x}{x+e^x} \ln x \end{aligned}$$

$$y' = \left(\frac{\ln(x+e^x)}{x} + \frac{1+e^x}{x+e^x} \ln x \right) (x+e^x)^{\ln x}$$

EXERCISES

Evaluate the derivatives of the following functions using chain rule:

7–1) $f(x) = \left(1 + x^4\right)^2$

7–2) $f(x) = e^{x^3}$

7–3) $f(x) = \ln\left(1 + x^2\right)$

7–4) $f(x) = \left(5 + x + 2x^3\right)^7$

7–5) $f(x) = \dfrac{x}{\sqrt{3x^2 + 2}}$

7–6) $f(x) = \dfrac{1}{(x^2 - 4x)^3}$

7–7) $f(x) = \left(\dfrac{2x}{x - 1}\right)^5$

7–8) $f(x) = \left(e^{3x} + 1\right)^5$

7–9) $f(x) = \sqrt{1 + \ln x}$

7–10) $f(x) = \sqrt{x^2 + 2e^{3x}}$

7–11) $f(x) = 4^{x^2 + 5x}$

7–12) $f(x) = xe^x \log_3\left(x + x^4\right)$

Find f'':

7–13) $f(x) = 5^{2x}$

7–14) $f(x) = \ln\left(3x\right)$

7–15) $f(x) = \sqrt{2 + x}$

7–16) $f(x) = x^7 e^{-x}$

Find f' using logarithmic differentiation:

7–17) $f(x) = \left(1 + 2x\right)^7\left(x^3 + 1\right)^4$

7–18) $f(x) = \dfrac{\left(3x^4 + x^2\right)^6}{\left(1 + x + x^2\right)^8}$

7–19) $f(x) = (\ln x)^x$

Find the equation of the line tangent to $f(x)$ at x_0:

7–20) $f(x) = 2x^2 - 8x + 4, \quad x_0 = 2.$

7–21) $f(x) = x\sqrt{2x + 4}, \quad x = 0.$

7–22) $f(x) = x^2(1 - x)^2, \quad x = 2.$

7–23) $f(x) = \dfrac{1}{1 + x^2}, \quad x = 0.$

Evaluate the derivatives of the following functions at the point $x = a$. In other words, find the value of $f'(a)$.

7–24) $f(x) = \dfrac{2x^2 - 3x + 12}{x}, \quad a = 2.$

7–25) $f(x) = x^{3/5}, \quad a = 32.$

7–26) $f(x) = \dfrac{5 + 3x^2}{8 + 4x}, \quad a = 0.$

7–27) $f(x) = \dfrac{\ln x}{x^4}, \quad a = 1.$

7–28) $f(x) = \big(1 + 2x\big)e^x, \quad a = 0.$

7–29) $f(x) = \sqrt{10 - e^{-x}}, \quad a = 0.$

7–30) $f(x) = \ln\left(\dfrac{x - 2}{3x - 3}\right), \quad a = 5.$

7–31) $f(x) = \left(2x + \dfrac{3}{x}\right)^2, \quad a = \dfrac{1}{2}.$

7–32) $f(x) = \big(4x + e^{5x}\big)^3, \quad a = 0.$

7–33) $f(x) = \dfrac{1}{2 + 4x + 8e^{2x}}, \quad a = 0.$

7–34) $f(x) = x \ln\sqrt{1 + 2x}, \quad a = 1.$

7–35) $f(x) = \ln\left(\dfrac{xe^x}{1 + x^2}\right), \quad a = 2.$

ANSWERS

7–1) $f'(x) = 8\big(1 + x^4\big)x^3$

7–2) $f'(x) = 3x^2 e^{x^3}$

7–3) $f'(x) = \dfrac{2x}{1 + x^2}$

7–4) $f'(x) = 7\big(5 + x + 2x^3\big)^6\big(1 + 6x^2\big)$

7–5) $f'(x) = \dfrac{2}{(3x^2 + 2)^{3/2}}$

7–6) $f'(x) = \dfrac{12 - 6x}{(x^2 - 4x)^4}$

7–7) $f'(x) = 5\left(\dfrac{2x}{x - 1}\right)^4 \dfrac{-2}{(x - 1)^2} = -\dfrac{160x^4}{(x - 1)^6}$

7–8) $f'(x) = 15(e^{3x} + 1)^4 e^{3x}$

7–9) $f'(x) = \dfrac{1}{2x\sqrt{1 + \ln x}}$

7–10) $f'(x) = \dfrac{2x + 6e^{3x}}{2\sqrt{x^2 + 2e^{3x}}}$

7–11) $f'(x) = 4^{x^2 + 5x}\,\big(2x + 5\big)$

7–12) $f'(x) = (e^x + xe^x)\log_3\big(x + x^4\big) + xe^x \dfrac{1 + 4x^3}{(x + x^4)\ln 3}$

7–13) $f''(x) = (4\ln^2 5)\,5^{2x}$

7–14) $f''(x) = -\dfrac{1}{x^2}$

7–15) $f''(x) = \dfrac{1}{4\,(2+x)^{3/2}}$

7–16) $f''(x) = \left(42x^5 - 14x^6 + x^7\right) e^{-x}$

7–17) $f'(x) = \left(1+2x\right)^7\left(x^3+1\right)^4\left[\dfrac{14}{1+2x} + \dfrac{12x^2}{x^3+1}\right]$

7–18) $f'(x) = \dfrac{\left(3x^4+x^2\right)^6}{\left(1+x+x^2\right)^8}\left[\dfrac{6(12x^3+2x)}{3x^4+x^2} - \dfrac{8(1+2x)}{1+x+x^2}\right]$

7–19) $f'(x) = (\ln x)^x\left[\ln\left(\ln x\right) + \dfrac{1}{\ln x}\right]$

7–20) $y = -4$

7–21) $y = 2x$

7–22) $y = -4x + 12$

7–23) $y = 1$

7–24) $f'(2) = -1$

7–25) $f'(32) = \dfrac{3}{20}$

7–26) $f'(0) = -\dfrac{5}{16}$

7–27) $f'(1) = 1$

7–28) $f'(0) = 3$

7–29) $f'(0) = \dfrac{1}{6}$

7–30) $f'(5) = \dfrac{1}{12}$

7–31) $f'\left(\dfrac{1}{2}\right) = -140$

7–32) $f'(0) = 27$

7–33) $f'(0) = -\dfrac{1}{5}$

7–34) $f'(1) = \dfrac{1}{2}\ln 3 + \dfrac{1}{3}$

7–35) $f'(-1) = \dfrac{7}{10}$

Chapter 8

Implicit Differentiation

An equation involving x and y may define y as a function of x. This is called an implicit function. For example, the following equations define y implicitly.

- $x^2 + y^2 = 1$,
- $ye^y + 2x - \ln y = 0$,
- $3xy + x^2y^3 + x = 5$,
- $e^x + e^y = \sqrt{x + 2y}$,

The following equations define y explicitly.

- $y = x^3 - 5x^2$,
- $y = \ln\left(x^2 - e^x\right)$,
- $y = x^3 + \sqrt{x} + xe^x$,
- $y = \dfrac{1}{1 + e^{x^2 - x}}$,

The derivative of y can be found without solving for y. This is called implicit differentiation. The main idea is:

- Differentiate with respect to x.
- Solve for y'.

Example 8–1: Find y' using the equation $y + y^3 = 3x^2 + 1$.

Solution: Find the derivative with respect to x:

$$y' + 3y^2y' = 6x$$

Therefore $y' = \dfrac{6x}{1 + 3y^2}$

Remark: Note that we are using chain rule here. For example, derivative of y^n is:

$$\begin{aligned} \frac{d(y^n)}{dx} &= \frac{d(y^n)}{dy}\,\frac{dy}{dx} \\ &= ny^{n-1}\,y' \end{aligned}$$

Example 8–2: Find the slope of the tangent line to the curve

$$x^2 + y^2 = 4 \quad \text{at the point} \quad \big(1,\ \sqrt{3}\big)$$

Solution: Let's differentiate both sides with respect to x:

$$\begin{aligned} x^2 + y^2 &= 4 \\ \frac{d}{dx}\Big(x^2 + y^2\Big) &= \frac{d}{dx}\Big(4\Big) \\ 2x + 2y\,\frac{dy}{dx} &= 0 \\ 2yy' &= -2x \\ y' &= -\frac{x}{y} \end{aligned}$$

Therefore at the point $\big(1,\ \sqrt{3}\big)$:

$$y' = -\frac{1}{\sqrt{3}}$$

An alternative method is to express y in terms of x explicitly as

$$y = \sqrt{4 - x^2}$$

and then differentiate as:

$$y' = \frac{1}{2}(4 - x^2)^{-1/2} \cdot (-2x)$$

and then insert $x = 1$, but usually this is not possible.

Example 8–3: Find y' using implicit differentiation where

$$xy + x^3y^2 = 5y$$

Solution: Let's differentiate both sides with respect to x. Note that we are also using product rule:

$$\begin{aligned} y + xy' + 3x^2y^2 + x^3 2yy' &= 5y' \\ xy' + 2x^3yy' - 5y' &= -y - 3x^2y^2 \\ \big(x + 2x^3y - 5\big)y' &= -y - 3x^2y^2 \\ y' &= -\frac{y + 3x^2y^2}{x + 2x^3y - 5} \end{aligned}$$

Example 8–4: Find y' using implicit differentiation where

$$x^2e^y + y = e^{3x}$$

Solution: Let's differentiate both sides with respect to x:

$$\begin{aligned} 2xe^y + x^2e^yy' + y' &= 3e^{3x} \\ \big(x^2e^y + 1\big)y' &= 3e^{3x} - 2xe^y \\ y' &= \frac{3e^{3x} - 2xe^y}{x^2e^y + 1} \end{aligned}$$

Example 8–5: Find y' using implicit differentiation where

$$\ln(x+3y) = \frac{1}{x^2}$$

Solution:
$$\begin{aligned}
\frac{1}{x+3y}\cdot(1+3y') &= \frac{-2}{x^3} \\
1+3y' &= \frac{-2(x+3y)}{x^3} \\
3y' &= -\frac{2}{x^2} - \frac{6y}{x^3} - 1 \\
y' &= -\frac{2}{3x^2} - \frac{2y}{x^3} - \frac{1}{3}
\end{aligned}$$

Example 8–6: Find y' using implicit differentiation where

$$ye^{xy} + x^4 \ln x = e^{3x}$$

Solution: First we use product rule, then chain rule:

$$y'e^{xy} + y\big(y+xy'\big)e^{xy} + 4x^3\ln x + x^3 = 3e^{3x}$$

$$e^{xy}y' + xye^{xy}y' = 3e^{3x} - 4x^3\ln x - x^3 - y^2e^{xy}$$

$$y' = \frac{3e^{3x} - 4x^3\ln x - x^3 - y^2e^{xy}}{e^{xy} + xye^{xy}}$$

$$y' = \frac{3e^{3x} - 4x^3\ln x - x^3 - y^2e^{xy}}{(1+xy)\,e^{xy}}$$

Example 8–7: Find the slope of the tangent line to the curve

$$x^8 + 4x^2y^2 + y^8 = 6 \quad \text{at the point} \quad (1,1)$$

Solution: Using implicit differentiation we obtain:

$$8x^7 + 8xy^2 + 8x^2yy' + 8y^7y' = 0$$

$$x^7 + xy^2 + \Big(x^2y + y^7\Big)y' = 0$$

$$\Rightarrow \quad y' = \frac{-x^7 - xy^2}{x^2y + y^7}$$

At $(1,1)$ the slope is: $y' = \dfrac{-2}{2} = -1.$

Example 8–8: Find y' at $(0,0)$ where

$$(1+x+2y)e^y + 3xe^x = 1 + x^2 + y^2$$

Solution: Using implicit differentiation we obtain:

$$(1+2y')e^y + (1+x+2y)e^yy' + 3e^x + 3xe^x = 2x + 2yy'$$

$$\big(2e^y + (1+x+2y)e^y - 2y\big)y' = 2x - e^y - 3e^x - 3xe^x$$

$$y' = \frac{2x - e^y - 3(1+x)e^x}{(3+x+2y)e^y - 2y}$$

$$y'(0,0) = -\frac{4}{3}.$$

EXERCISES

Find y' using implicit differentiation:

8–1) $x^2y^3 + 3xy^2 + y = 5$

8–2) $xye^x + (x + 2y)^2 = x$

8–3) $(x^2 + y)^2 = y^3$

8–4) $x = y + y^{2/3}$

8–5) $(1 + e^{-x})^2 = \ln(x + y)$

8–6) $\ln y = y^3 + \ln x$

8–7) $e^{xy} = x + 2y$

8–8) $x^2 + \ln y = 3xy$

Find y' using implicit differentiation:

8–9) $x^2y = e^y$

8–10) $x^4 + y^4 = 3x + 5y$

8–11) $xy^2 = 1 + \ln(xy)$

8–12) $e^y + x^2e^x = 18$

8–13) $y^2 \ln y = x^3e^x$

8–14) $\sqrt{5x + y^3} + xy = 12$

8–15) $\dfrac{2}{x} + \dfrac{7}{y} = 9$

8–16) $x^{1/3} + y^{1/5} = y$

Find y' at the indicated point using implicit differentiation:

8–17) $(1 + 2x + 3y)^2 = 13x \ln x + 7y^5 + 29$ at $(1, 1)$

8–18) $3x - 2y + 8x^2 + 5y^2 + 9e^{9x} + 7e^{2y} = 16$ at $(0, 0)$

8–19) $x^2y^2 + 2xy^3 + y - 10x + 11 = 0$ at $(2, 1)$

8–20) $xy^4 + 3y^5 + x - 3y^3 = 0$ at $(0, 1)$

8–21) $2x - 4y^4 + x^2y^6 + 11y^3 = 0$ at $(3, -1)$

8–22) $\sqrt{11 + y^2} - 12xy + 2y^2 + 4x = 0$ at $(1, 5)$

8–23) $xe^x - ye^y + xy - 1 = 0$ at $(1, 1)$

8–24) $\ln(xy) + xy^2 - \ln 3x - 6y = 0$ at $(2, 3)$

ANSWERS

8–1) $y' = -\dfrac{2xy^3 + 3y^2}{3x^2y^2 + 6xy + 1}$

8–2) $y' = -\dfrac{ye^x + xye^x + 2x + 4y - 1}{xe^x + 4x + 8y}$

8–3) $y' = \dfrac{4x^3 + 4xy}{3y^2 - 2x^2 - 2y}$

8–4) $y' = \dfrac{1}{1 + \frac{2}{3}y^{-1/3}}$

8–5) $y' = -2(x + y)e^{-x}(1 + e^{-x}) - 1$

8–6) $y' = \dfrac{y}{x(1 - 3y^3)}$

8–7) $y' = \dfrac{1 - ye^{xy}}{xe^{xy} - 2}$

8–8) $y' = \dfrac{3y^2 - 2xy}{1 - 3xy}$

8–9) $y' = \dfrac{2xy}{e^y - x^2}$

8–10) $y' = \dfrac{4x^3 - 3}{5 - 4y^4}$

8–11) $y' = \dfrac{y - xy^3}{2x^2y^2 - x}$

8–12) $y' = -(2x + x^2)e^{x-y}$

8–13) $y' = \dfrac{3x^2e^x + x^3e^x}{2y\ln y + y}$

8–14) $y' = -\dfrac{2y\sqrt{5x + y^3} + 5}{2x\sqrt{5x + y^3} + 3y^2}$

8–15) $y' = -\dfrac{2y^2}{7x^2}$

8–16) $y' = \dfrac{5y^{4/5}}{3x^{2/3}\left(5y^{4/5} - 1\right)}$

8–17) $y'\Big|_{(1,1)} = -11$

8–18) $y'\Big|_{(0,0)} = -7$

8–19) $y'\Big|_{(2,1)} = \dfrac{4}{21}$

8–20) $y'\Big|_{(0,1)} = -\dfrac{1}{3}$

8–21) $y'\Big|_{(3,-1)} = \dfrac{8}{5}$

8–22) $y'\Big|_{(1,5)} = \dfrac{336}{53}$

8–23) $y'\Big|_{(1,1)} = \dfrac{2e + 1}{2e - 1}$

8–24) $y'\Big|_{(2,3)} = -\dfrac{27}{19}$

Chapter 9

L'Hôpital's Rule

Some limits like $\frac{0}{0}, \quad \frac{\infty}{\infty}, \ldots$ etc. are called indeterminate forms. These limits may turn out to be definite numbers, or infinity, or may not exist. Note that when we say $\frac{0}{0}$ we do not mean dividing the number 0 by the number 0. This would be undefined.

$\frac{0}{0}$ is a notation we use to denote the limit

$$\lim_{x \to a} \frac{f}{g}$$

where $\lim_{x \to a} f = 0, \quad \lim_{x \to a} g = 0$. The case $\frac{\infty}{\infty}$ is similar to this.

For example, consider the limits

$$\lim_{x \to 0} \frac{x^5}{x^2}, \qquad \lim_{x \to 0} \frac{x^5}{x^7}, \qquad \lim_{x \to 0} \frac{3x^5}{4x^5}$$

All are of the form $\frac{0}{0}$ but their results are $0, \infty$ and $\frac{3}{4}$.

Consider the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

Assume that this is an indeterminate form of the type:

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

Suppose $g'(x) \neq 0$ on an open interval containing a (except possibly at $x = a$).

Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if this limit exists, or is $\pm\infty$.

This is called the **L'Hôpital's Rule**.

Example 9–1: Evaluate the following limit: (if it exists.)

$$\lim_{x \to 0} \frac{e^x - 1}{2x}$$

Solution: This limit is in the form $\frac{0}{0}$, so we will use L'Hôpital's rule:

$$\lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2}$$

To evaluate this limit, insert $x = 0$ to obtain:

$$\lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$$

Example 9–2: Evaluate the following limit: (if it exists.)

$$\lim_{x \to 0} \frac{e^{3x} - e^x - 2x}{x^2}$$

Solution: Limit is in the form $\frac{0}{0}$ $\Rightarrow$ use L'Hôpital's rule:

$$\lim_{x \to 0} \frac{e^{3x} - e^x - 2x}{x^2} = \lim_{x \to 0} \frac{3e^{3x} - e^x - 2}{2x}$$

This second limit is also in the form $\frac{0}{0}$, so we will use L'Hôpital's rule once more:

$$\lim_{x \to 0} \frac{3e^{3x} - e^x - 2}{2x} = \lim_{x \to 0} \frac{9e^{3x} - e^x}{2}$$

Now just insert $x = 0$:

$$\lim_{x \to 0} \frac{9e^{3x} - e^x}{2} = 4$$

Example 9–3: Evaluate the limit $\lim_{x \to \infty} \frac{e^x}{x^3}$.

Solution: Indeterminacy of the form $\frac{\infty}{\infty}$ $\Rightarrow$ use L'Hôpital.

$$\begin{aligned} \lim_{x \to \infty} \frac{e^x}{x^3} &= \lim_{x \to \infty} \frac{e^x}{3x^2} \\ &= \lim_{x \to \infty} \frac{e^x}{6x} \\ &= \lim_{x \to \infty} \frac{e^x}{6} \\ &= \infty \end{aligned}$$

The result would be the same if it were x^{30} rather than x^3. Exponential function increases faster than all polynomials.

Example 9–4: Evaluate the limit $\lim_{x \to \infty} \frac{\ln x}{x^2}$.

Solution: Indeterminacy of the form $\frac{\infty}{\infty}$ $\Rightarrow$ use L'Hôpital.

$$\begin{aligned} \lim_{x \to \infty} \frac{\ln x}{x^2} &= \lim_{x \to \infty} \frac{\frac{1}{x}}{2x} \\ &= \lim_{x \to \infty} \frac{1}{2x^2} \\ &= 0 \end{aligned}$$

The result would be 0 for any x^k. Logarithmic function increases slower than all polynomials.

Example 9–5: Evaluate the limit $\lim\limits_{x\to 1} \dfrac{x^{10}-1}{x^7-1}$.

Solution: It is possible to solve this question using the algebraic identities:

$$x^{10}-1=\big(x-1\big)\big(x^9+x^8+\cdots+x+1\big)$$

$$x^7-1=\big(x-1\big)\big(x^6+x^5+\cdots+x+1\big)$$

$$\begin{aligned}\lim_{x\to 1}\frac{x^{10}-1}{x^7-1} &= \lim_{x\to 1}\frac{x^9+x^8+\cdots+x+1}{x^6+x^5+\cdots+x+1}\\ &= \frac{10}{7}\end{aligned}$$

but this is too complicated. Limit is in the form $\dfrac{0}{0}$ and using L'Hôpital gives the same result easily.

$$\begin{aligned}\lim_{x\to 1}\frac{x^{10}-1}{x^7-1} &= \lim_{x\to 1}\frac{10x^9}{7x^6}\\ &= \frac{10}{7}\end{aligned}$$

Example 9–6: Evaluate the limit $\lim\limits_{x\to 0} \dfrac{(1+x)^{4/3}-1}{x}$.

Solution: Indeterminacy of the form $\dfrac{0}{0}$ $\Rightarrow$ Use L'Hôpital:

$$\begin{aligned}\lim_{x\to 0}\frac{(1+x)^{4/3}-1}{x} &= \lim_{x\to 0}\frac{\frac{4}{3}(1+x)^{1/3}}{1}\\ &= \frac{4}{3}\end{aligned}$$

Example 9–7: Evaluate the following limit: (if it exists.)

$$\lim_{x\to 0}\frac{e^x-1-x-\frac{x^2}{2}}{x^4}$$

Solution: This limit is in the form $\dfrac{0}{0}$, so using L'Hôpital's rule we obtain:

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1-x-\frac{x^2}{2}}{x^4} &= \lim_{x\to 0}\frac{e^x-1-x}{4x^3}\\ &= \lim_{x\to 0}\frac{e^x-1}{12x^2}\\ &= \lim_{x\to 0}\frac{e^x}{24x}\end{aligned}$$

At this point, the limit is NOT in the form $\dfrac{0}{0}$, so we can NOT use L'Hôpital. Checking the numerator and denominator, we see that:

$$=\infty$$

Example 9–8: Evaluate the limit $\lim\limits_{x\to\infty} \dfrac{\ln x+x^2}{xe^x}$.

Solution: Indeterminacy of the form $\dfrac{\infty}{\infty}$ $\Rightarrow$ Use L'Hôpital:

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln x+x^2}{xe^x} &= \lim_{x\to\infty}\frac{\frac{1}{x}+2x}{e^x+xe^x}\\ &= \lim_{x\to\infty}\frac{-\frac{1}{x^2}+2}{e^x+e^x+xe^x}\\ &= 0\end{aligned}$$

EXERCISES

Evaluate the following limits (if they exist):

9–1) $\displaystyle\lim_{x\to 0} \frac{e^{5x} - e^{4x} - x}{x^2}$

9–2) $\displaystyle\lim_{x\to 0} \frac{e^{3x} - 1}{\ln(x+1)}$

9–3) $\displaystyle\lim_{x\to 0} \frac{e^{x} - 1}{e^{2x} + 3x - 1}$

9–4) $\displaystyle\lim_{x\to \infty} \frac{3x^2 + 4\ln x}{6x^2 + 7\ln x}$

9–5) $\displaystyle\lim_{x\to \infty} \frac{2e^{x} + 5x}{7e^{x} + 8x + 12}$

9–6) $\displaystyle\lim_{x\to \infty} \frac{\ln(x + x^4)}{x}$

9–7) $\displaystyle\lim_{x\to 64} \frac{x^{1/3} - 4}{x^{1/2} - 8}$

9–8) $\displaystyle\lim_{x\to 0} \frac{\sqrt{9+2x} - 3}{\sqrt{16+x} - 4}$

Evaluate the following limits (if they exist):

9–9) $\displaystyle\lim_{x\to 2} \frac{x^3 - 2x^2 + 6x - 12}{x^3 - 2x^2 + 8x - 16}$

9–10) $\displaystyle\lim_{x\to e} \frac{\ln x - 1}{x - e}$

9–11) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x}$

9–12) $\displaystyle\lim_{x\to 3} \frac{x^3 - 4x - 15}{x^2 + x - 12}$

9–13) $\displaystyle\lim_{x\to \infty} \frac{\ln x}{\sqrt[3]{x}}$

9–14) $\displaystyle\lim_{x\to 1} \frac{x^6 - 1}{x^4 - 1}$

9–15) $\displaystyle\lim_{x\to 3} \frac{e^{x} - e^{3}}{x^2 - 9}$

9–16) $\displaystyle\lim_{x\to 0} \frac{4^{x} - 1}{2^{x} - 1}$

Evaluate the following limits (if they exist):

9–17) $\lim\limits_{x\to 2} \dfrac{\ln \frac{x}{2}}{x(x-2)}$

9–18) $\lim\limits_{x\to 0} \dfrac{\sqrt{a+bx}-\sqrt{a+cx}}{x}$

9–19) $\lim\limits_{x\to 0} \dfrac{(1+x)^k-1-kx}{x^2}$

9–20) $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1}$

9–21) $\lim\limits_{x\to 1} \dfrac{\ln x - x + 1}{(x-1)^2}$

9–22) $\lim\limits_{x\to 1/2} \dfrac{\ln(2x)}{2x^2+x-1}$

9–23) $\lim\limits_{x\to \infty} x^3 e^{-x}$

9–24) $\lim\limits_{x\to 0^+} x \ln x$

ANSWERS

9–1) $\frac{9}{2}$

9–2) 3

9–3) $\frac{1}{5}$

9–4) $\frac{1}{2}$

9–5) $\frac{2}{7}$

9–6) 0

9–7) $\frac{1}{3}$

9–8) $\frac{8}{3}$

9–9) $\frac{5}{6}$

9–10) $\frac{1}{e}$

9–11) $\frac{1}{2}$

9–12) $\frac{23}{7}$

9–13) 0

9–14) $\frac{3}{2}$

9–15) $\frac{e^3}{6}$

9–16) 2

9–17) $\frac{1}{4}$

9–18) $\frac{b-c}{2\sqrt{a}}$

9–19) $\frac{k(k-1)}{2}$

9–20) 1

9–21) $-\frac{1}{2}$

9–22) $\frac{2}{3}$

9–23) 0

9–24) 0

Chapter 10

Finding Maximum and Minimum Values

Local and Absolute Extrema:
Extremum is either minimum or maximum. Extrema is the plural form.

Absolute Extrema: If

$$f(c) \leqslant f(x)$$

for all x on a set S of real numbers, $f(c)$ is the absolute minimum value of f on S.

Similarly if

$$f(c) \geqslant f(x)$$

for all x on S, $f(c)$ is the absolute maximum value of f on S.

Local Extrema: $f(c)$ is local minimum if

$$f(c) \leqslant f(x)$$

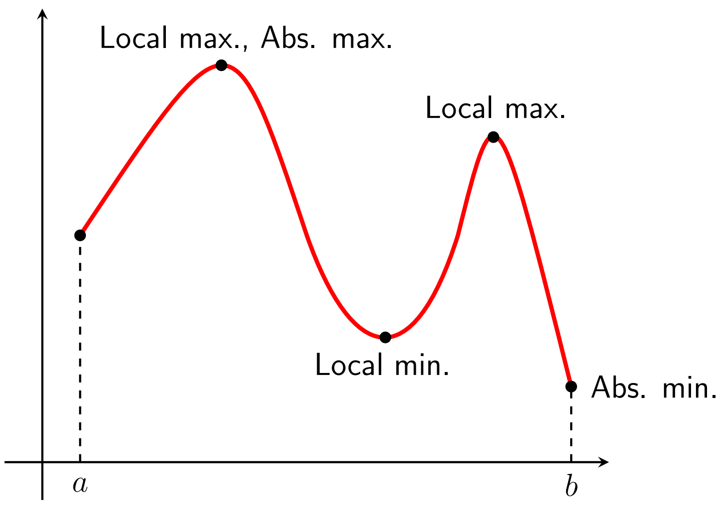
for all x in some open interval containing c.

Similarly, $f(c)$ is local maximum if

$$f(c) \geqslant f(x)$$

for all x in some open interval containing c.

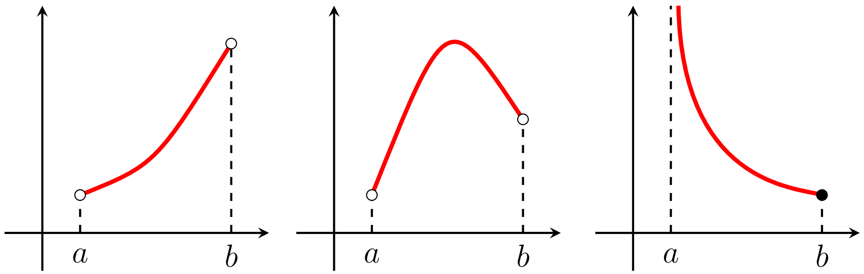
Local extrema are points that are higher (or lower) than the points around them.



As you can see in the figure, a point can be both local and absolute extremum. Also, it may be an absolute extremum without being a local one or vice versa.

Question: Does a continuous function always have an absolute maximum and an absolute minimum value?

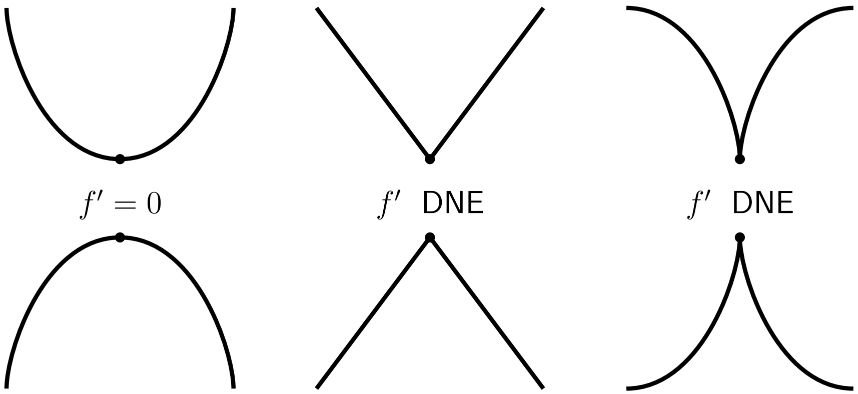
This depends on the interval. It may or may not have such values on an open interval.



No max. or min. Max. but no min. Min. but no max.

Theorem: If the function f is continuous on the closed interval $[a, b]$, then f has a maximum and a minimum value on $[a, b]$.

Critical Point: A number c is called a critical point of the function f if $f'(c) = 0$ or $f'(c)$ does not exist.



The main ideas about extremum points can be summarized as:

1. f can have local extremum only at a critical point.
2. f can have absolute extremum only at a critical point or an endpoint.

For example, the local extremum point of the parabola

$$f(x) = ax^2 + bx + c$$

will be at the point $x = -\dfrac{b}{2a}$ (called the vertex) because this is the point where the derivative is zero:

$$f'(x) = 2ax + b = 0 \quad \Rightarrow \quad x = -\frac{b}{2a}$$

If $a > 0$ it is a minimum and if $a < 0$ it is a maximum.

The local minimum point of the function $f(x) = |ax + b|$ will be at the point $x = -\dfrac{b}{a}$ because this is the point where the derivative is undefined.

How to find absolute extrema:

- Find the points where $f' = 0$.
- Find the points where f' does not exist.
- Consider such points only if they are **inside** the given interval.
- Consider endpoints.
- Check all candidates. Both absolute minimum and maximum are among them.

Example 10–1: Find the maximum and minimum values of

$$f(x) = -x^2 + 10x + 2$$

on the interval $[1,\,4]$.

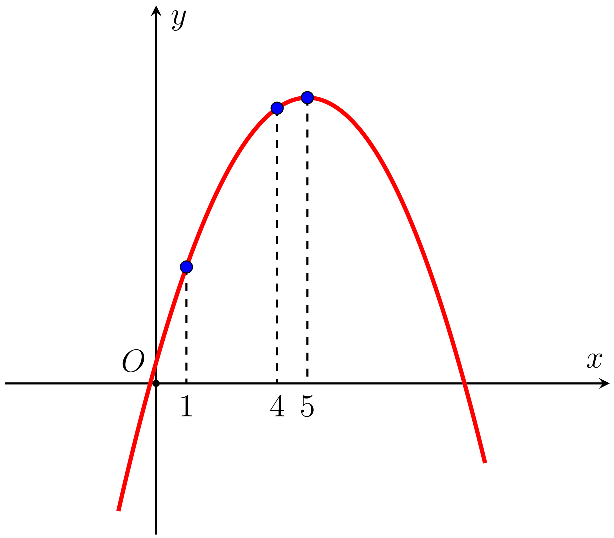
Solution: Let's find the critical points first:

$$f' = -2x + 10 = 0$$

$\Rightarrow \quad x = 5$ is the only critical point. But it is not in our interval $[0,\,4]$, so our candidates for extrema are the endpoints:

x	$f(x)$
1	11
4	26

Clearly, absolute minimum is 11 and it occurs at $x = 1$. Absolute maximum is 26 and it occurs at $x = 4$.



Example 10–2: Find the maximum and minimum values of

$$f(x) = -x^2 + 10x + 2$$

on the interval $[2,\,10]$.

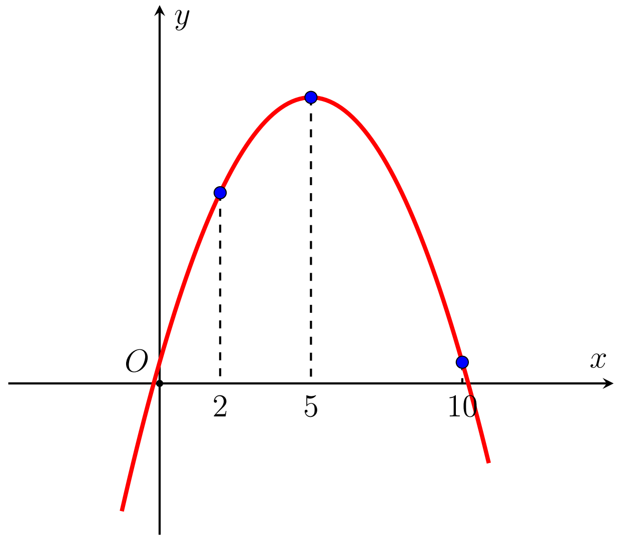
Solution: Although it is the same function, interval is different.

$$f'(x) = -2x + 10 = 0$$

$\Rightarrow \quad x = 5$ is the only critical point. It is inside the interval.

x	$f(x)$
2	18
5	27
10	2

Absolute minimum is 2 and it occurs at $x = 10$. Absolute maximum is 27 and it occurs at $x = 5$.



Example 10–3: Find the maximum and minimum values of

$$f(x) = \left|x - 8\right|$$

on the interval $\left[6,\, 12\right]$.

Solution: Let's write the function in piecewise defined form:

$$f(x) = \begin{cases} -x+8 & \text{if} \quad x < 8 \\ x-8 & \text{if} \quad x \geqslant 8 \end{cases}$$

Derivative is:

$$f'(x) = \begin{cases} -1 & \text{if} \quad x < 8 \\ 1 & \text{if} \quad x > 8 \end{cases}$$

Derivative is never zero. The only critical point is $x = 8$. Derivative does not exist at that point.

Now we need a table that shows all critical points in the interval and endpoints:

x	$f(x)$
6	2
8	0
12	4

We can see that absolute minimum is 0 and absolute maximum is 4.

Example 10–4: Find the maximum and minimum values of

$$f(x) = \left|16 - x^2\right|$$

on the interval $\left[-3,\, 5\right]$.

Solution: First, express f as a piecewise defined function:

$$f(x) = \begin{cases} x^2-16 & \text{if} \quad x < -4 \\ 16-x^2 & \text{if} \quad -4 \leqslant x \leqslant 4 \\ x^2-16 & \text{if} \quad x > 4 \end{cases}$$

Derivative is:

$$f'(x) = \begin{cases} 2x & \text{if} \quad x < -4 \\ -2x & \text{if} \quad -4 < x < 4 \\ 2x & \text{if} \quad x > 4 \end{cases}$$

f' is zero at $x = 0$ and it is undefined at $x = \pm 4$. We will not consider $x = -4$ because it is outside the interval. So, critical points in the interval are:

$$x = 0, \quad x = 4$$

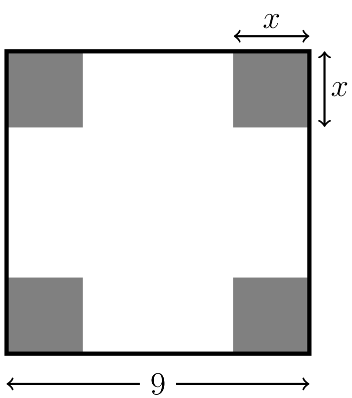
Together with the endpoints, we can make the following table:

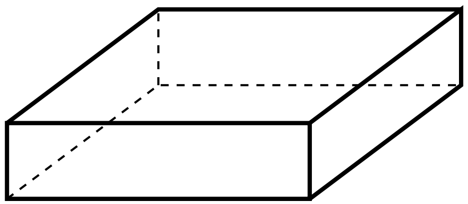
x	$f(x)$	
-3	7	
0	16	Abs. Max.
4	0	Abs. Min.
5	9	

Applied Optimization:
Finding the maximum or minimum of a function has many real-life applications. For these problems:

- Express the quantity to be maximized or minimized as a function of the independent variable. (We will call it x)
- Determine the interval over which x changes.
- Solve the problem in the usual way. (Find the critical points, check the function at critical points and endpoints)

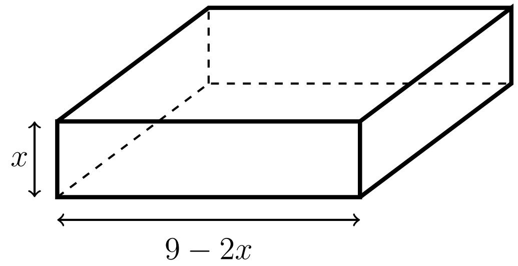
Example 10–5: A piece of cardboard is shaped as a 9×9 square. We will cut four small squares from the corners and make an open top box. What is the maximum possible volume of the box?





Solution: If the squares have edge length x, we can express the volume as:

$$V(x) = x(9-2x)^2 = 81x - 36x^2 + 4x^3$$



Considering the maximum and minimum possible values, we can see that $x \in \left[0, \frac{9}{2}\right]$. Now we can use maximization procedure:

$$V'(x) = 81 - 72x + 12x^2 = 0$$

$$27 - 24x + 4x^2 = 0$$

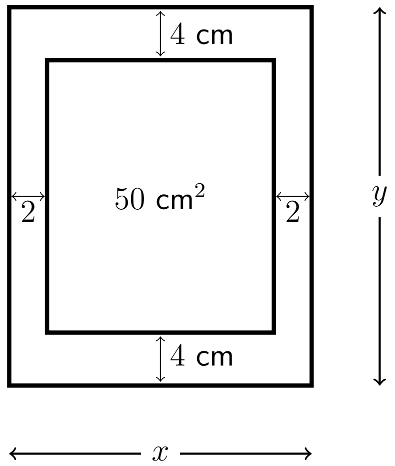
$$(2x-9)(2x-3) = 0$$

$$x = \frac{9}{2} \quad \text{or} \quad x = \frac{3}{2}$$

Checking all critical and endpoints, we find that $x = \dfrac{3}{2}$ gives the maximum volume, which is:

$$V = 54.$$

Example 10–6: You are designing a rectangular poster to contain 50 cm^2 of picture area with a 4 cm margin at the top and bottom and a 2 cm margin at each side. Find the dimensions x and y that will minimize the total area of the poster.



Solution: $(x-4)(y-8)=50 \quad \Rightarrow \quad y=\dfrac{50}{x-4}+8$

$$A = xy = x\left(\frac{50}{x-4}+8\right)$$

$$A' = \frac{50}{x-4}+8-\frac{50x}{(x-4)^2}=0$$

$$\frac{200}{(x-4)^2}=8 \quad \Rightarrow \quad (x-4)^2=25$$

$$\Rightarrow \quad x=9 \quad \text{and} \quad y=18.$$

Example 10–7: You are selling tickets for a concert. If the price of a ticket is $\$15$, you expect to sell 600 tickets. Market research reveals that, sales will increase by 40 for each $\$0.5$ price decrease, and decrease by 40 for each $\$0.5$ price increase. For example, at $\$14.5$ you will sell 640 tickets. At $\$16$ you will sell 520 tickets.

What should the ticket price be for largest possible revenue?

Solution: We need to define our terms first:

- x denotes the sale price of a ticket in $\$$,
- N denotes the number of tickets sold,
- R denotes the revenue.

According to market research, $N = 600 + 40\,\dfrac{15-x}{0.5}$.
In other words:

$N = 600 + 80\big(15-x\big) = 1800 - 80x.$

Note that we sell zero tickets if $x = \dfrac{1800}{80} = 22.5$.
(That's the highest possible price.)

Revenue is:
$$\begin{aligned} R &= Nx \\ &= (1800-80x)x \\ &= 1800x - 80x^2 \end{aligned}$$

This is a maximization problem where the interval of the variable is: $x \in \big[0,\ 22.5\big]$.

$R' = 1800 - 160x = 0 \quad \Rightarrow \quad x = 11.25$

Checking the critical point $x = 11.25$ and endpoints 0 and 22.5 we see that the maximum revenue occurs at $x = 11.25$.

Example 10–8: A helicopter will cover a distance of 235 km. with constant speed v km/h. The amount of fuel used during flight in terms of liters per hour is

$$75 + \frac{v}{3} + \frac{v^2}{1200}.$$

Find the speed v that minimizes total fuel used during flight.

Solution: The time it takes for flight is: $t = \dfrac{235}{v}$

The total amount of fuel consumed is:

$$\left(75 + \frac{v}{3} + \frac{v^2}{1200}\right) \cdot t = 235 \cdot \left(\frac{75}{v} + \frac{1}{3} + \frac{v}{1200}\right)$$

In other words we have to find v that minimizes $f(v)$ on $v \in \big(0, \infty\big)$ where:

$$f(v) = \frac{75}{v} + \frac{1}{3} + \frac{v}{1200}$$

Note that the distance 235 km. is not relevant. Once we find the optimum speed, it is optimum for all distances.

$$f'(v) = -\frac{75}{v^2} + \frac{1}{1200} = 0$$

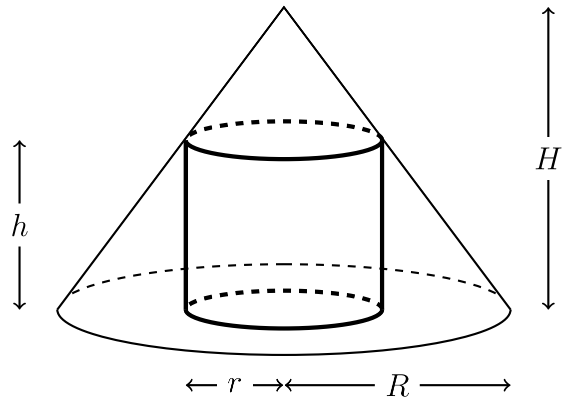
$$\Rightarrow \quad v^2 = 75 \cdot 1200 = 90\,000$$

$$\Rightarrow \quad v = 300$$

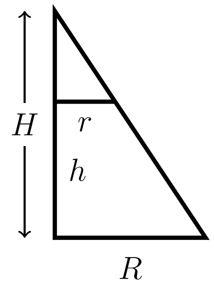
This value clearly gives the minimum, because:

$$\lim_{v\to 0} f = \lim_{v\to\infty} f = \infty.$$

Example 10–9: A cylinder is inscribed in a cone of radius R, height H. What is the maximum possible the volume of the cylinder?



Solution:



$$V = \pi r^2 h$$

Similar triangles: $\dfrac{H-h}{H} = \dfrac{r}{R}$

$$\Rightarrow \quad h = H\left(1 - \frac{r}{R}\right)$$

$$V = \pi r^2 H\left(1 - \frac{r}{R}\right) = \pi H\left(r^2 - \frac{r^3}{R}\right)$$

$$\frac{dV}{dr} = \pi H\left(2r - \frac{3r^2}{R}\right) = 0$$

$$2r = \frac{3r^2}{R} \quad \Rightarrow \quad r = \frac{2R}{3} \quad \Rightarrow \quad h = \frac{H}{3}$$

Maximum Volume: $V = \dfrac{4}{27}\pi R^2 H.$

EXERCISES

Find the absolute maximum and minimum values of $f(x)$ on the given interval:

10–1) $f(x) = x^{\frac{2}{3}}$ on $[-2, 3]$

10–2) $f(x) = 10x(2 - \ln x)$ on $[1, e^2]$

10–3) $f(x) = 12 - x^2$ on $[2, 4]$

10–4) $f(x) = 12 - x^2$ on $[-2, 4]$

10–5) $f(x) = 3x^3 - 16x$ on $[-2, 1]$

10–6) $f(x) = x + \dfrac{9}{x}$ on $[1, 4]$

10–7) $f(x) = 3x^5 - 5x^3$ on $[-2, 2]$

10–8) $f(x) = |3x - 5|$ on $[0, 2]$

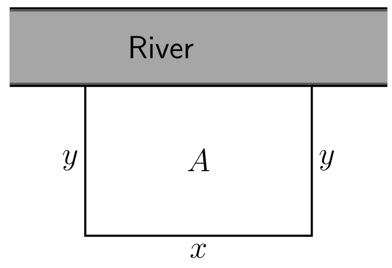
10–9) $f(x) = |x^2 + 6x - 7|$ on $[-8, 2]$

10–10) $f(x) = x\sqrt{1 - x^2}$ on $[-1, 1]$

10–11) $f(x) = e^{-x^2}$ on $[-1, 2]$

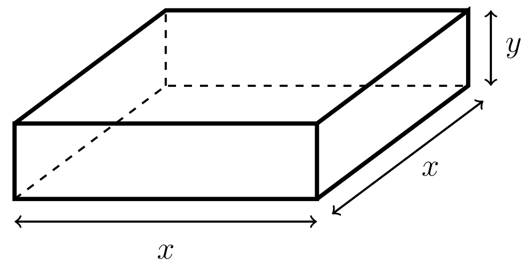
10–12) $f(x) = \dfrac{120}{\sqrt{x}}$ on $[16, 36]$

10–13) We will cover a rectangular area with a 36m-long fence. The area is near a river so we will only cover the three sides. Find the maximum possible area.



$$y + x + y = 36$$

10–14) An open top box has volume 75 cm^3 and is shaped as seen in the figure. Material for base costs 12\$/cm^2 and material for sides costs 10\$/cm^2. Find the dimensions x and y that give the minimum total cost.



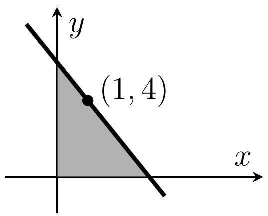
10–15) Find the dimensions of the right circular cylinder of the greatest volume if the surface area is 54π.

10–16) What is the maximum possible area of the rectangle with its base on the $x-$axis and its two upper vertices are on the graph of $y = 4 - x^2$?

10–17) Find the shortest distance between the point $(2, 0)$ and the curve $y = \sqrt{x}$.

10–18) Find the point on the line $y = ax + b$ that is closest to origin.

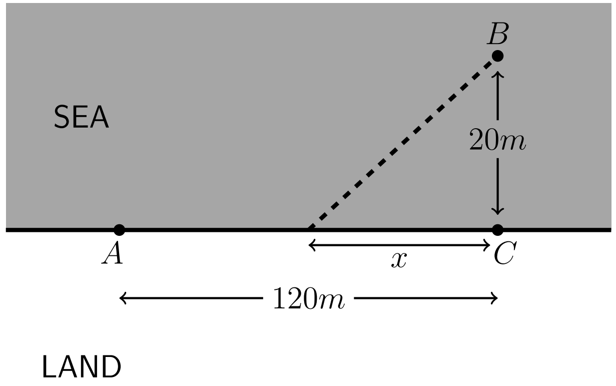
10–19) We choose a line passing through the point $(1, 4)$ and find the area in the first quadrant bounded by the line and the coordinate axes. What line makes this area minimum?



10–20) Two vertical poles are 21 meters apart. Their heights are 12m and 16m. A cable is stretched from the top of first pole to a point on the ground and then to the top of the second pole. Find the minimum possible length of the cable.

10–21) Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius R.

10–22) A swimmer is drowning on point B. You are at point A. You may run up to point C and then swim, or you may start swimming a distance x earlier. Assume your running speed is 5 m/s and your swimming speed is 3 m/s. What is the ideal x?



10–23) A coffee chain has 20 shops in a city. Average daily profit per shop is $\$3000$. Each new shop decreases the average profit of all shops by $\$100$. For example, if the company opens 3 new shops, average profit becomes $\$2700$.

What is the ideal number of shops, assuming the company wants to maximize total profit?

10–24) A $500-$room hotel's nightly rent is $\$80$ and it is full every night. For each $\$1$ increase in rent, 5 fewer rooms are rented. For example, if rent is $\$100$ there are 400 full and 100 empty rooms. The cost of service per room (for full rooms) is $\$40$ per day. What is the nightly rent that maximizes profit? What is the maximum profit?

ANSWERS

10–1) Absolute Minimum: 0, Absolute Maximum: $\sqrt[3]{9}$.

10–2) Absolute Minimum: 0, Absolute Maximum: $10e$.

10–3) Absolute Minimum: -4, Absolute Maximum: 8.

10–4) Absolute Minimum: -4, Absolute Maximum: 12.

10–5) Absolute Minimum: -13, Absolute Maximum: $\frac{128}{9}$.

10–6) Absolute Minimum: 6, Absolute Maximum: 10.

10–7) Absolute Minimum: -56, Absolute Maximum: 56.

10–8) Absolute Minimum: 0, Absolute Maximum: 5.

10–9) Absolute Minimum: 0, Absolute Maximum: 16.

10–10) Absolute Minimum: $-\frac{1}{2}$, Absolute Maximum: $\frac{1}{2}$.

10–11) Absolute Minimum: $\frac{1}{e^2}$, Absolute Maximum: 1.

10–12) Absolute Minimum: 20, Absolute Maximum: 30.

10–13) $A = 162$

10–14) $x = 5, \quad y = 3$

10–15) $r = 3, \quad h = 6$

10–16) $\frac{32}{3\sqrt{3}}$

10–17) $\frac{\sqrt{7}}{2}$

10–18) $\left(\frac{-ab}{1+a^2}, \frac{b}{1+a^2}\right)$

10–19) $y = -4x + 8$

10–20) $35m$

10–21) $r = \frac{\sqrt{2}}{\sqrt{3}}R, \quad h = \frac{2}{\sqrt{3}}R$

10–22) $x = 15m$

10–23) 25

10–24) $110, \quad 24\,500$

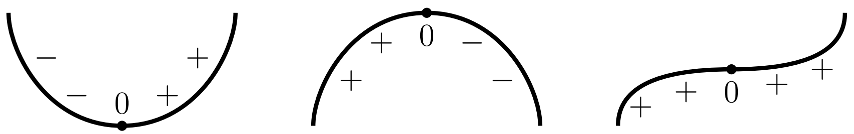
Chapter 11

Curve Sketching

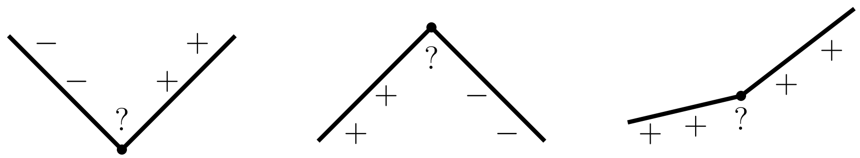
First Derivative Test: At a critical point, the derivative is zero or undefined. Let f be a continuous function and let $x = c$ be a critical point of it. Suppose f' exists in some interval containing c except possibly at c.

f has a local extremum at c if and only if f' changes sign at c.

- Sign change: $-$ to $+$ $\quad\Rightarrow\quad$ $f(c)$ is a local minimum.
- Sign change: $+$ to $-$ $\quad\Rightarrow\quad$ $f(c)$ is a local maximum.



Local Min. $\qquad$ Local Max. $\qquad$ Neither



Example 11–1: Find the intervals where $f(x) = 2x^3 - 9x^2 + 5$ is increasing and decreasing and local extrema of this function.

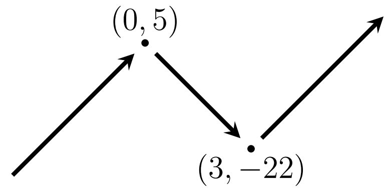
Solution: $f'(x) = 6x^2 - 18x = 0 \quad\Rightarrow\quad x = 0$ or $x = 3$.

There are two critical points, 0 and 3. Note that $f(0) = 5$ and $f(3) = -22$.

x changes sign at 0 and $(x - 3)$ changes sign at 3. We can find the sign of $x(x - 3)$ by multiplying these signs.

x		0		3	
$(x-3)$	$-$		$-$	0	$+$
$f' = x(x-3)$	$+$	0	$-$	0	$+$
f	increasing		decreasing		increasing

Based on this table, we can see that the graph is roughly like this:



Therefore $(0, 5)$ is local maximum and $(3, -22)$ is local minimum.

Concavity: The graph of a differentiable function is concave up if f' increasing, it is concave down if f' decreasing.

Test for Concavity:

- If $f''(x) > 0$, then f is concave up at x.
- If $f''(x) < 0$, then f is concave down at x.

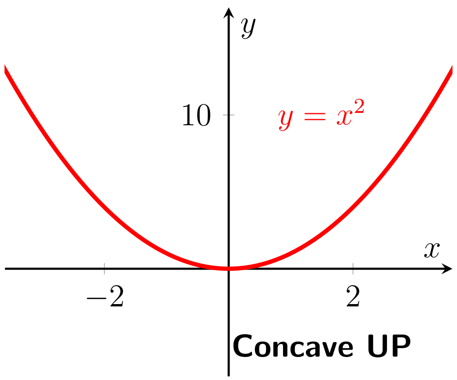
Inflection Point: An inflection point is a point where the concavity changes. In other words, if:

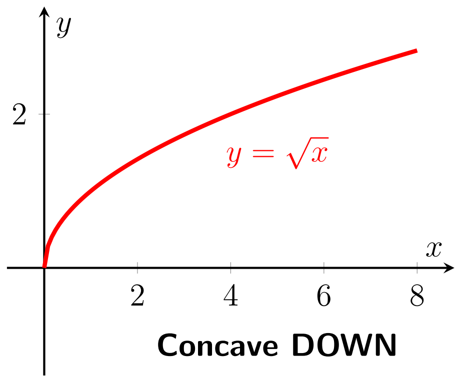
- f is continuous at $x = a$,
- $f'' > 0$ on the left of a and $f'' < 0$ on the right, or vice versa.

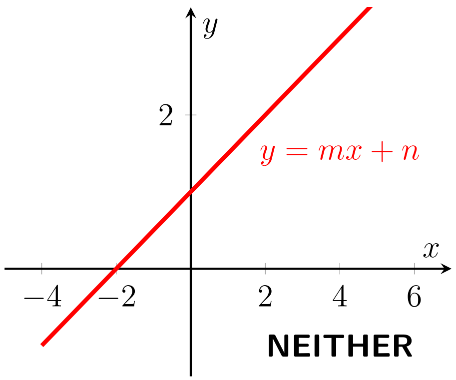
then $x = a$ is an inflection point.

This means either $f''(a) = 0$ or $f''(a)$ does not exist.

Examples:







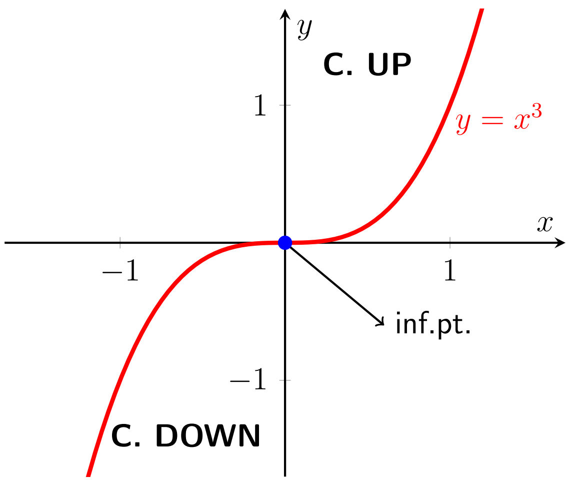
Example 11–2: Determine the concavity of $f(x) = x^3$. Find inflection points. (If there is any.)

Solution:

$$\begin{aligned} f &= x^3 \\ f' &= 3x^2 \\ f'' &= 6x \end{aligned}$$

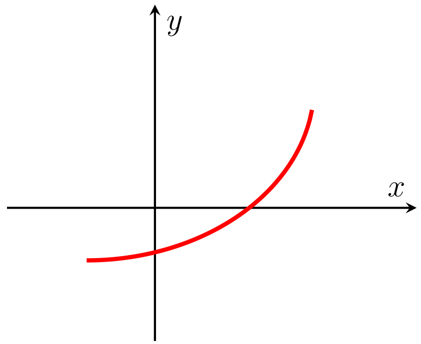
- For $x > 0$, $f'' > 0 \quad \Rightarrow \quad f$ is concave up.
- For $x < 0$, $f'' < 0 \quad \Rightarrow \quad f$ is concave down.
- $x = 0$ is the inflection point.

x		0	
f''	$-$	0	$+$
f is:	concave down		concave up



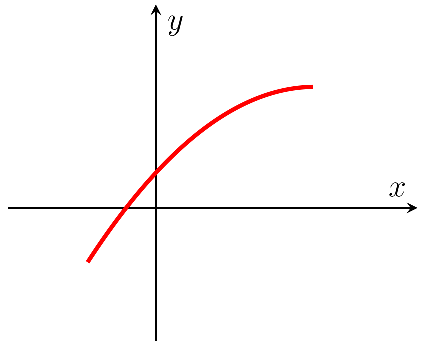
Shape of a graph based on first and second derivatives:

$f' > 0, \quad f'' > 0$



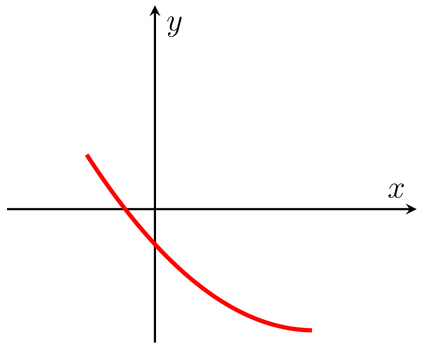
Increasing, Concave up.

$f' > 0, \quad f'' < 0$



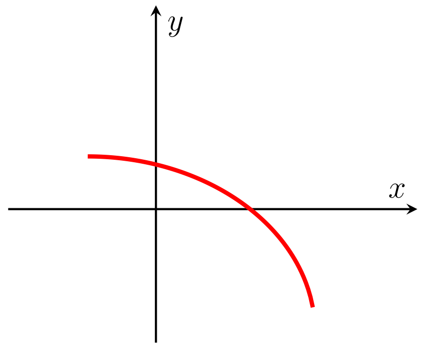
Increasing, Concave down.

$f' < 0, \quad f'' > 0$



Decreasing, Concave up.

$f' < 0, \quad f'' < 0$



Decreasing, Concave down.

Curve Sketching:

- Identify domain of f, symmetries, x and y intercepts. (if any)
- Find first and second derivatives of f.
- Find critical points, inflections points.
- Make a table and include all this information.
- Sketch the curve using the table.

Example 11–3: Sketch the graph of $f(x) = x^3 + 3x^2 - 24x$.

Solution: $\lim\limits_{x\to\infty} f = +\infty, \quad \lim\limits_{x\to-\infty} f = -\infty$

$$\begin{aligned} f' &= 3x^2 + 6x - 24 \\ &= 3(x+4)(x-2) \end{aligned}$$

$f' = 0 \quad \Rightarrow \quad x = -4,$ and $x = 2.$
These are the critical points.

$f'' = 6x + 6 = 0 \quad \Rightarrow \quad x = -1.$
This is the inflection point.

Some specific points on the graph are:

$f(-4) = 80, \quad f(-1) = 26.$

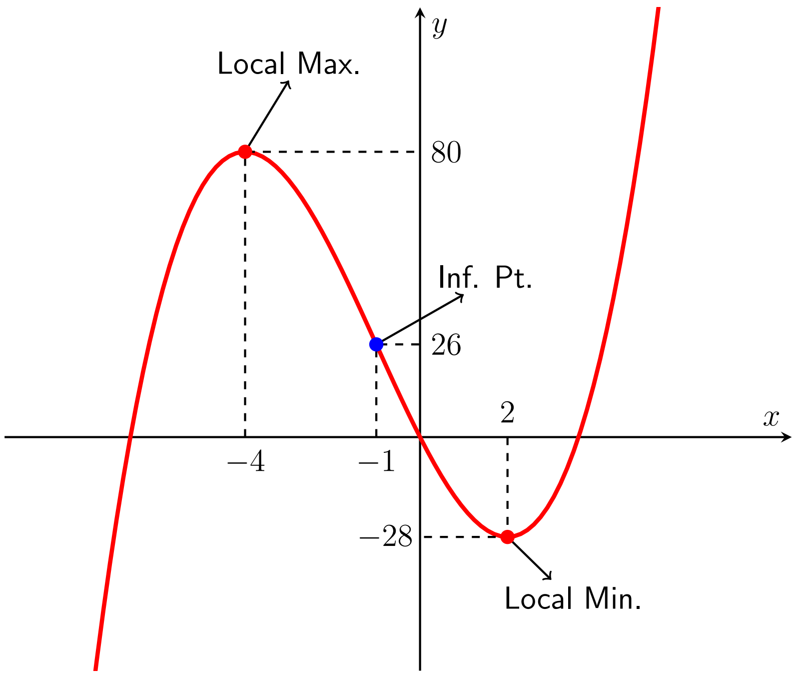
$f(0) = 0, \quad f(2) = -28.$

The equation $f(x) = 0$ gives $x = 0$ or $x^2 + 3x - 24 = 0$ in other words $x = \dfrac{-3 \pm \sqrt{105}}{2}$. Using a calculator we find $x_1 = -6.6, \; x_2 = 3.6$ but it is possible to sketch the graph without

these points. Putting all this information on a table, we obtain:

x		-4		-1		2	
f'	$+$	0	$-$		$-$	0	$+$
f''	$-$		$-$	0	$+$		$+$
f	$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

Based on this table, we can sketch the graph as:



EXERCISES

Determine the intervals where the following functions are increasing and decreasing:

11–1) $f(x) = x^3 - 12x - 5$

11–2) $f(x) = 16 - 4x^2$

11–3) $f(x) = \dfrac{1}{(x-4)^2}$

11–4) $f(x) = \dfrac{x^2 - 3}{x - 2}$

11–5) $f(x) = 4x^5 + 5x^4 - 40x^3$

11–6) $f(x) = x^4 e^{-x}$

11–7) $f(x) = \dfrac{\ln x}{x}$

11–8) $f(x) = 5x^6 + 6x^5 - 45x^4$

11–9) $f(x) = x^4 - 2x^2 + 1$

11–10) $f(x) = \dfrac{x}{x+1}$

Identify local maxima, minima and inflection points, then sketch the graphs of the following functions:

11–11) $f(x) = x^3 - 3x^2 - 9x + 11$

11–12) $f(x) = -2x^3 + 21x^2 - 60x$

11–13) $f(x) = 3x^4 + 4x^3 - 36x^2$

11–14) $f(x) = (x-1)^2(x+2)^3$

11–15) $f(x) = x^6 - 6x^5$

11–16) $f(x) = x^3 e^{-x}$

11–17) $f(x) = e^{-x^2}$

11–18) $f(x) = \dfrac{x}{x^2+1}$

11–19) $f(x) = x \ln |x|$

11–20) $f(x) = -x^4 + 32x^2$

ANSWERS

11–1) Increasing on $(-\infty,\ -2)$, decreasing on $(-2,\ 2)$, increasing on $(2,\ \infty)$.

11–2) Increasing on $(-\infty,\ 0)$, decreasing on $(0,\ \infty)$.

11–3) Increasing on $(-\infty,\ 4)$, decreasing on $(4,\ \infty)$.

11–4) Increasing on $(-\infty,\ 1)$, decreasing on $(1,\ 2) \cup (2,\ 3)$, increasing on $(3,\ \infty)$.

11–5) Increasing on $(-\infty,\ -3)$, decreasing on $(-3,\ 0) \cup (0,\ 2)$, increasing on $(2,\ \infty)$.

11–6) Decreasing on $(-\infty,\ 0)$, increasing on $(0,\ 4)$, decreasing on $(4,\ \infty)$.

11–7) Increasing on $(0,\ e)$, decreasing on $(e,\ \infty)$.

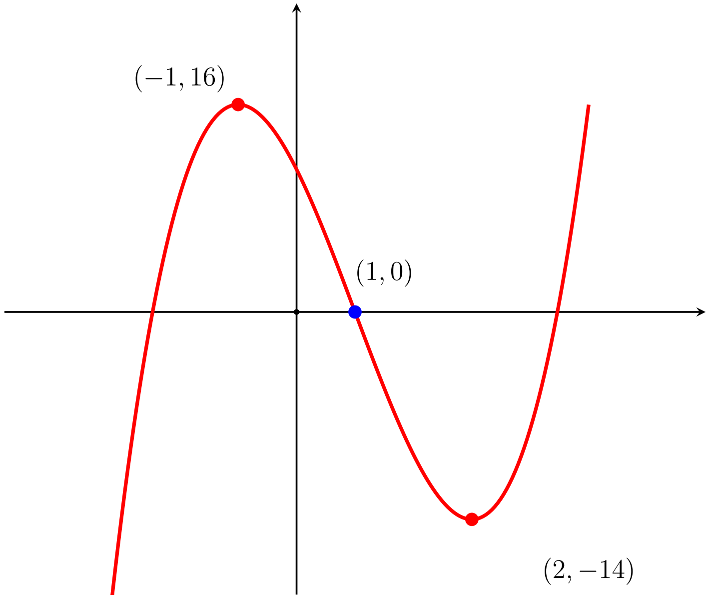
11–8) Decreasing on $(-\infty,\ -3)$, increasing on $(-3,\ 0)$, decreasing on $(0,\ 2)$, increasing on $(2,\ \infty)$.

11–9) Decreasing on $(-\infty,\ -1)$, increasing on $(-1,\ 0)$, decreasing on $(0,\ 1)$, increasing on $(1,\ \infty)$.

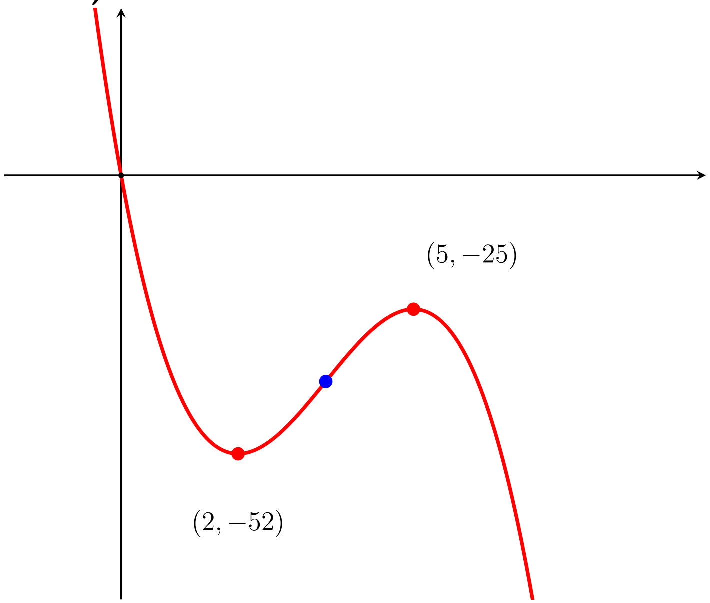
11–10) Increasing on $(-\infty,\ -1) \cup (-1,\ \infty)$.

(Blue dots denote inflection points, red dots local extrema.)

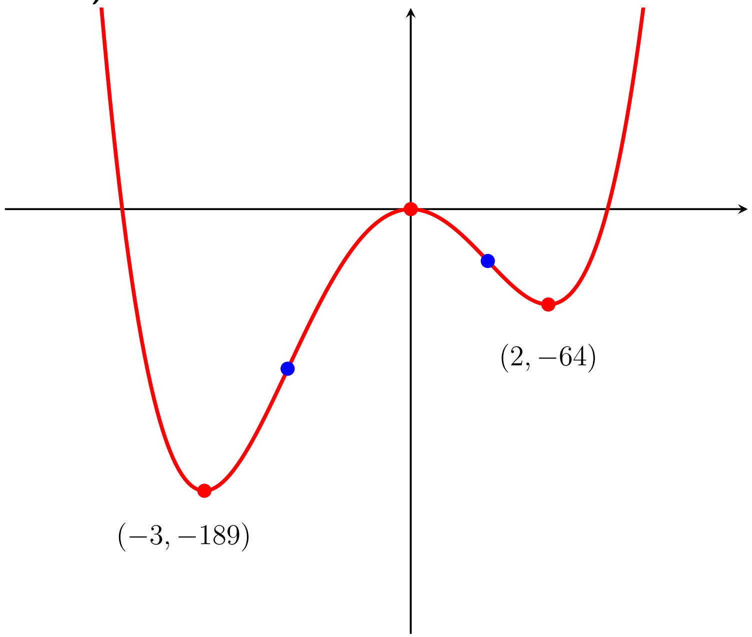
11–11)



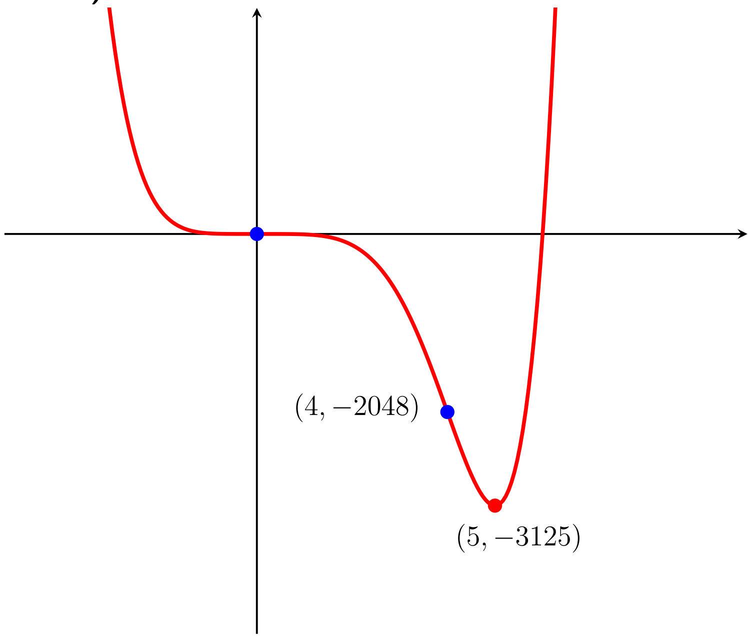
11–12)



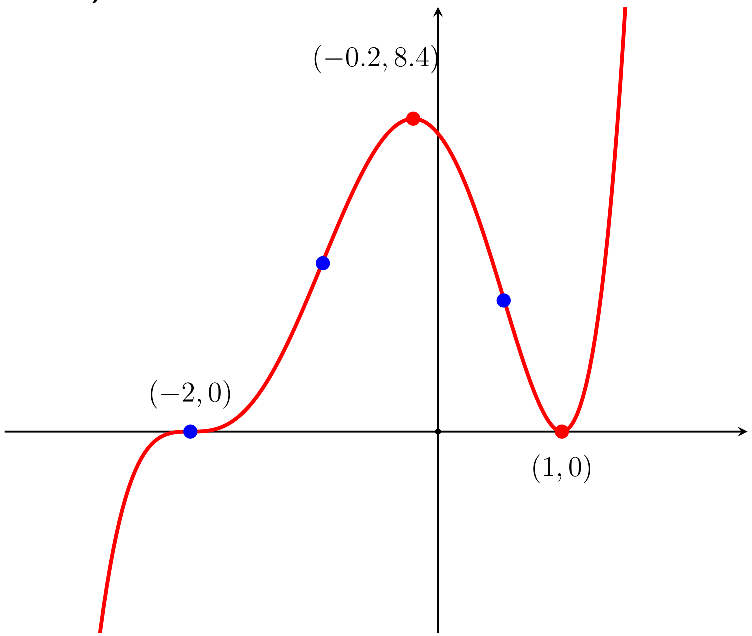
11–13)



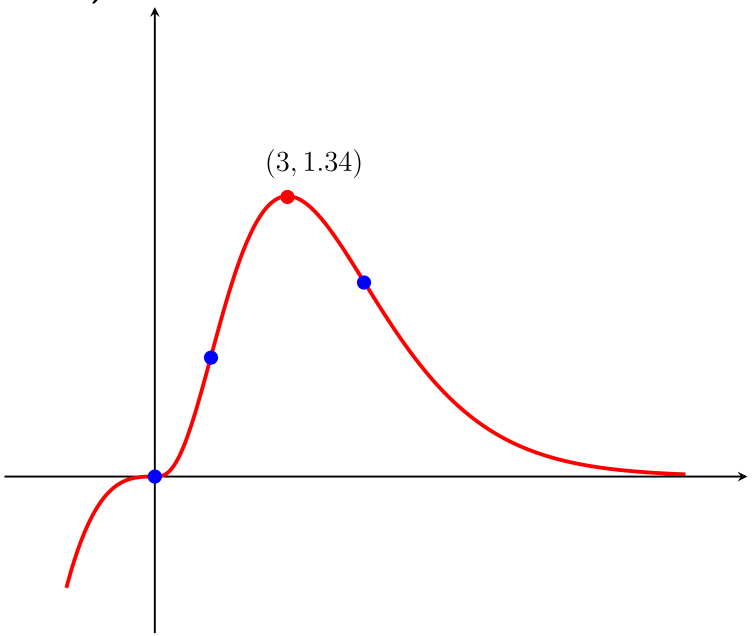
11–15)



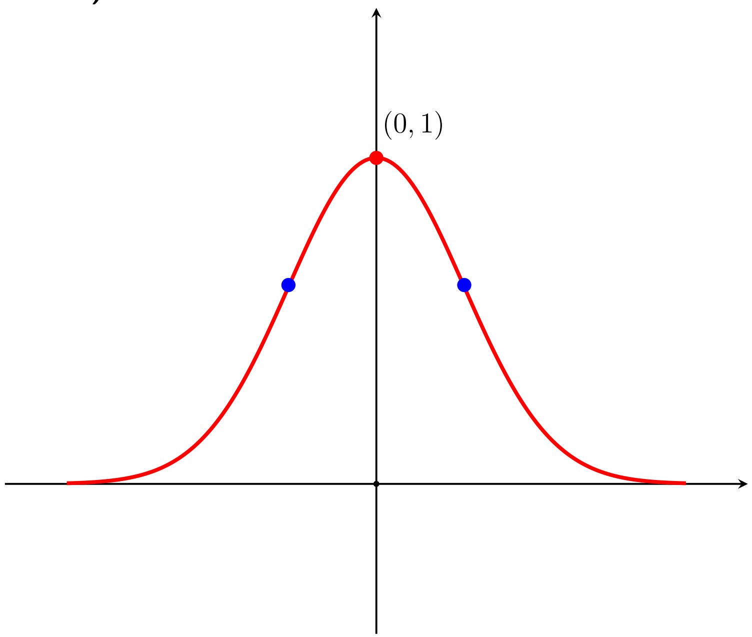
11–14)



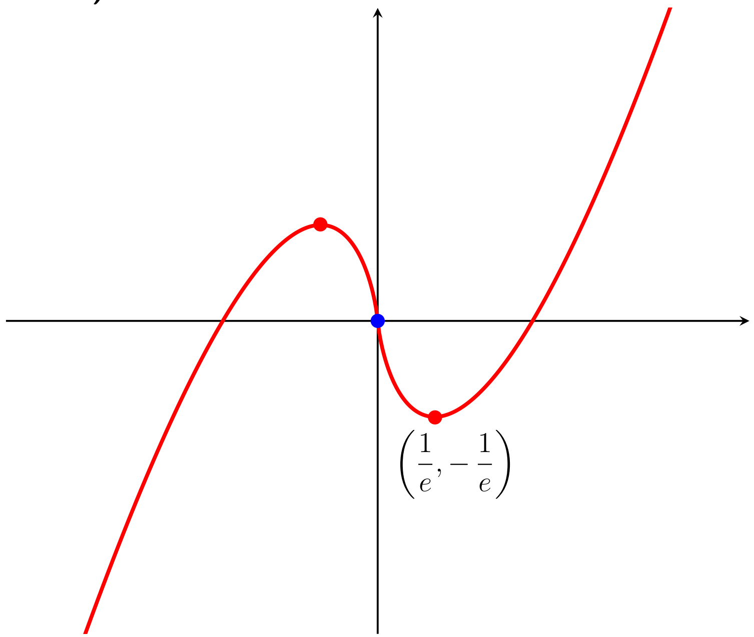
11–16)



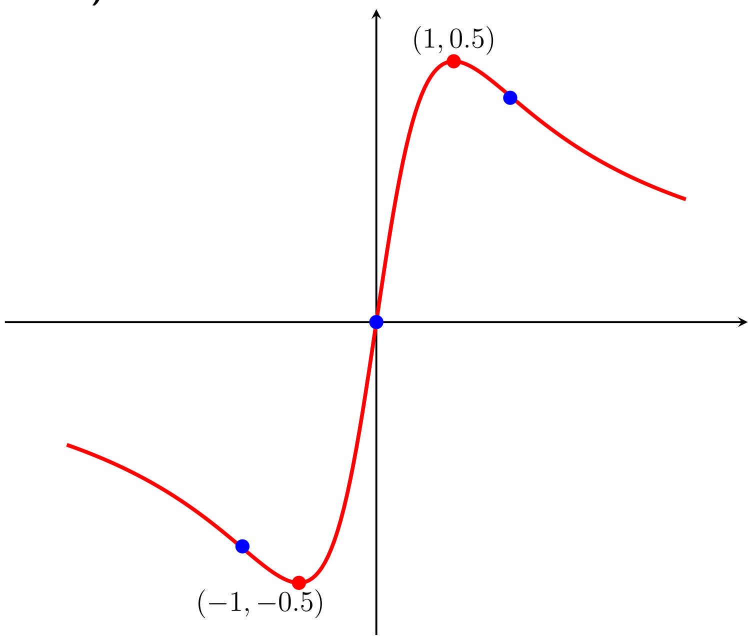
11–17)



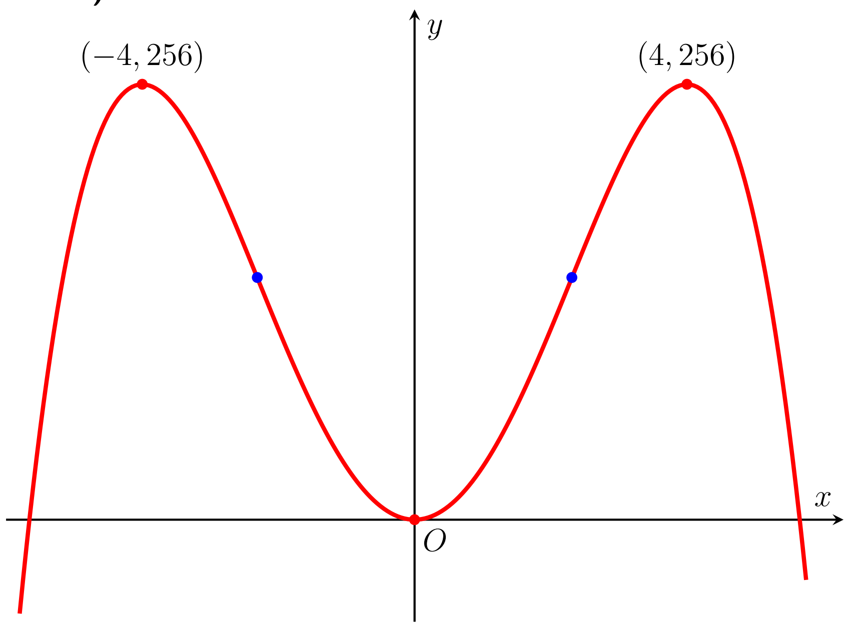
11–19)



11–18)



11–20)



Chapter 12

Integrals

Indefinite Integrals:
If f is the derivative of F, then F is the antiderivative of f.

$$F'(x) = f(x)$$

For example, the antiderivative of $f(x) = x$ is:

$$F(x) = \frac{x^2}{2}, \qquad \text{or} \quad F(x) = \frac{x^2}{2} + 5, \qquad \text{or} \quad F = \frac{x^2}{2} - 7$$

Note that all of the above functions have the property

$$F'(x) = x = f(x)$$

In other words, the constant numbers we add do not matter. There are infinitely many such antiderivatives. The collection of all antiderivatives of f is called the indefinite integral of f.

$$\int f(x)\,dx = F(x) + c$$

Here, c is an arbitrary constant.

Using the fact that integral and derivative are inverse operations, we obtain:

$$\begin{aligned}
\int 1\,dx &= x + c \\
\int x\,dx &= \frac{x^2}{2} + c \\
\int x^k\,dx &= \frac{x^{k+1}}{k+1} + c, \quad k \neq -1 \\
\int \frac{1}{x}\,dx &= \ln|x| + c \\
\int e^x\,dx &= e^x + c \\
\int e^{ax}\,dx &= \frac{e^{ax}}{a} + c
\end{aligned}$$

Example 12–1: Evaluate the integral $\int \left(\frac{1}{x^3} - 2x + 4\right) dx$

Solution:

$$\begin{aligned}
\int \left(\frac{1}{x^3} - 2x + 4\right) dx &= \int \frac{dx}{x^3} - \int 2x\,dx + \int 4\,dx \\
&= \int x^{-3}\,dx - 2\int x\,dx + \int 4\,dx \\
&= \frac{x^{-2}}{-2} - x^2 + 4x + c \\
&= -\frac{1}{2x^2} - x^2 + 4x + c
\end{aligned}$$

Example 12–2: Evaluate the integral $\int \frac{x^2 - 1}{x\sqrt{x}}\,dx$

Solution:

$$\begin{aligned}
\int \frac{x^2 - 1}{x\sqrt{x}}\,dx &= \int \frac{x^2}{x\sqrt{x}}\,dx - \int \frac{1}{x\sqrt{x}}\,dx \\
&= \int x^{\frac{1}{2}}\,dx - \int x^{-\frac{3}{2}}\,dx \\
&= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c \\
&= \frac{2}{3}x\sqrt{x} + \frac{2}{\sqrt{x}} + c
\end{aligned}$$

Example 12–3: Find a function $f(x)$ such that $f'(x) = 5e^x$ and $f(0) = 9$.

Solution: We have to integrate $5e^x$ to find $f(x)$:

$$\int 5e^x\,dx = 5e^x + c$$

Now, let's use the fact that $f(0) = 9$ to determine c:

$$5e^0 + c = 9 \quad\Rightarrow\quad 5 + c = 9 \quad\Rightarrow\quad c = 4$$

$$f(x) = 5e^x + 4.$$

Example 12–4: Find a function $f(x)$ such that $f''(x) = 4 - \frac{8}{x^2}$ and $f(1) = -15, \quad f'(1) = 7$.

Solution: Let's integrate $4 - \frac{8}{x^2}$ to find $f'(x)$:

$$f'(x) = \int f''(x)\,dx = \int \left(4 - \frac{8}{x^2}\right) dx = 4x + \frac{8}{x} + c_1$$

Using $f'(1) = 7$ we find: $4 + 8 + c_1 = 7$

$$\Rightarrow \quad c_1 = -5, \quad f'(x) = 4x + \frac{8}{x} - 5.$$

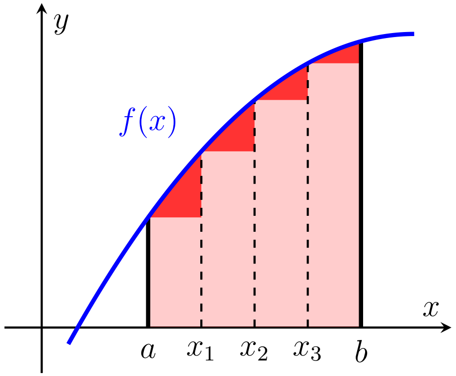
$$\begin{aligned}
f(x) &= \int f'(x)\,dx \\
&= \int \left(4x + \frac{8}{x} - 5\right) dx = 2x^2 + 8\ln|x| - 5x + c_2
\end{aligned}$$

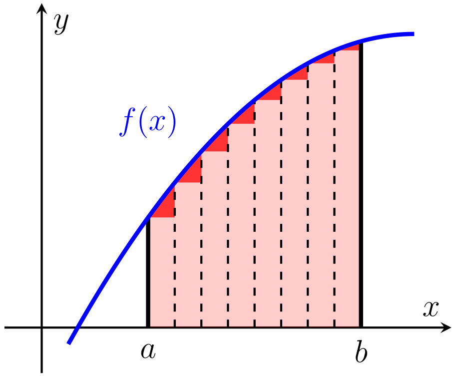
Using $f(1) = -15$ we find:

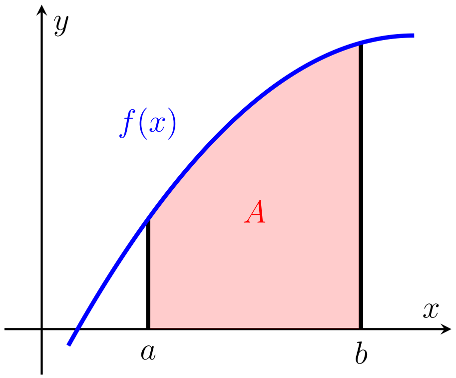
$$2 + 0 - 5 + c_2 = -15 \quad\Rightarrow\quad c_2 = -12.$$

$$\Rightarrow \quad f(x) = 2x^2 + 8\ln|x| - 5x - 12.$$

Definite Integrals: We denote the area under a curve using definite integrals. We can approximate the area under the graph of f between $x = a$ and $x = b$ using rectangles.







This area is denoted by:

$$A = \int_a^b f(x)dx$$

Note that definite integral is a number. (Not a function.)

Definite Integral Properties:

$$\begin{aligned}
\int_a^a f(x)dx &= 0 \\
\int_a^b cf(x)dx &= c\int_a^b f(x)dx \\
\int_a^b f(x) \pm g(x)dx &= \int_a^b f(x)dx \pm \int_a^b g(x)dx \\
\int_a^b f(x)dx &= -\int_b^a f(x)dx \\
\int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx
\end{aligned}$$

$$(\text{Min } f)\cdot(b-a) \leqslant \int_a^b f(x)dx \leqslant (\text{Max } f)\cdot(b-a)$$

We can see all these by using simple geometric rules.

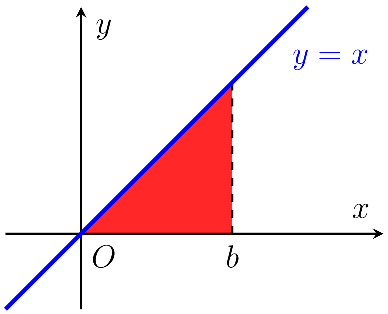
The Fundamental Theorem of Calculus: It seems that we have two different concepts of integral. One is inverse operation of derivative, the other is area under a curve. But there is a very close relationship between the two:

Let f be a continuous function on the interval $[a,\, b]$. If F is any anti-derivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

This is called the fundamental theorem of calculus. We will use the notation $F(x)\Big|_a^b$ for $F(b) - F(a)$.

For example, suppose we want to find the following area:



$$\begin{aligned}\int_0^b x\,dx &= \frac{x^2}{2}\Big|_0^b \\ &= \frac{b^2}{2} - 0 \\ &= \frac{b^2}{2}\end{aligned}$$

Example 12–5: Evaluate $\displaystyle\int_0^2 \big(3x^2 + 8x - 5\big)\,dx$

Solution:
$$\begin{aligned}\int_0^2 3x^2 + 8x - 5\,dx &= x^3 + 4x^2 - 5x\,\Big|_0^2 \\ &= 2^3 + 4\cdot 2^2 - 5\cdot 2 - 0 \\ &= 14\end{aligned}$$

Example 12–6: Evaluate $\displaystyle\int_1^9 \frac{5}{\sqrt{x}}\,dx$

Solution:
$$\begin{aligned}\int_1^9 \frac{5}{\sqrt{x}}\,dx &= \int_1^9 5x^{-\frac{1}{2}}\,dx \\ &= \frac{5\,x^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_1^9 \\ &= 5\cdot 2\cdot 9^{\frac{1}{2}} - 5\cdot 2\cdot 1^{\frac{1}{2}} \\ &= 30 - 10 = 20\end{aligned}$$

Example 12–7: Evaluate $\displaystyle\int_3^7 \frac{dx}{x}$

Solution:
$$\begin{aligned}\int_3^7 \frac{dx}{x} &= \ln|x|\,\Big|_3^7 \\ &= \ln 7 - \ln 3 = \ln\frac{7}{3}.\end{aligned}$$

EXERCISES

Evaluate the following indefinite integrals:

12–1) $\int (x^2 + 3x^5)\, dx$

12–2) $\int (1 + 10x - 24x^7)\, dx$

12–3) $\int \frac{1}{7x^4}\, dx$

12–4) $\int -5e^x\, dx$

12–5) $\int \frac{2}{x}\, dx$

12–6) $\int (x^4 - x\sqrt{x} + 2x)\, dx$

12–7) $\int (x^2 + 4)(x - 5)\, dx$

12–8) $\int \frac{1}{\sqrt[3]{x^5}}\, dx$

12–9) $\int \frac{2u^2 - 3u + 12}{u^2}\, du$

12–10) $\int \frac{3}{e^{-z}}\, dz$

Evaluate the following definite integrals:

12–11) $\int_0^1 \sqrt{x^3}\, dx$

12–12) $\int_1^{27} x^{-1/3}\, dx$

12–13) $\int_1^3 \frac{5}{u}\, du$

12–14) $\int_2^3 (x^2 + 5x - 1)\, dx$

12–15) $\int_0^1 e^x\, dx$

12–16) $\int_{-1}^2 (1 + 4e^x)\, dx$

12–17) $\int_1^4 5t^{-2}\, dt$

12–18) $\int_1^9 \frac{1 - \sqrt{x}}{\sqrt{x}}\, dx$

12–19) $\int_1^{32} x^{-2/5}\, dx$

12–20) $\int_{-2}^{-1} \frac{1}{x^3}\, dx$

ANSWERS

12–1) $\dfrac{x^3}{3} + \dfrac{x^6}{2} + c$

12–2) $x + 5x^2 - 3x^8 + c$

12–3) $-\dfrac{1}{21x^3} + c$

12–4) $-5e^x + c$

12–5) $2\ln|x| + c$

12–6) $\dfrac{x^5}{5} - \dfrac{2x^{5/2}}{5} + x^2 + c$

12–7) $\dfrac{x^4}{4} - \dfrac{5x^3}{3} + 2x^2 - 20x + c$

12–8) $-\dfrac{3}{2}x^{-2/3} + c$

12–9) $2u - 3\ln|u| - 12u^{-1} + c$

12–10) $3e^z + c$

12–11) $\dfrac{2}{5}$

12–12) 12

12–13) $5\ln 3$

12–14) $\dfrac{107}{6}$

12–15) $e - 1$

12–16) $3 + 4(e^2 - e^{-1})$

12–17) $\dfrac{15}{4}$

12–18) -4

12–19) $\dfrac{35}{3}$

12–20) $-\dfrac{3}{8}$

Chapter 13

Substitution

Using the chain rule, we obtain:

$$\frac{d}{dx} F\big(u(x)\big) = \frac{dF(u)}{du} \cdot \frac{du(x)}{dx}$$

If we integrate both sides, we see that

$$\int f\big(u(x)\big)\, u'(x)\, dx = F\big(u(x)\big) + c$$

where $f = F'$, or more simply

$$\int f\big(u(x)\big)\, u'(x)\, dx = \int f(u)\, du$$

Here, the idea is to make a substitution that will simplify the given integral. For example, the choice $u = x^2 + 1$ simplifies the integral:

$$\int \frac{2x\, dx}{x^2+1} \quad \to \quad \int \frac{du}{u}$$

Note that here we replace both x and dx:

$$\begin{aligned} x^2 + 1 &\to u \\ 2x\, dx &\to du \end{aligned}$$

Example 13–1: Evaluate the integral $\displaystyle\int \big(x^4+1\big)^2\, 4x^3\, dx$.

Solution: Let's use the new variable $u = x^4 + 1$. In that case

$$\big(x^4+1\big)^2 = u^2$$

Also,

$$\frac{du}{dx} = 4x^3 \quad \Rightarrow \quad du = 4x^3\, dx$$

If we write the integral using the new variable:

$$\begin{aligned} I &= \int u^2\, du \\ &= \frac{u^3}{3} + c \end{aligned}$$

But we have to express this in terms of the original variable:

$$I = \frac{\big(x^4+1\big)^3}{3} + c$$

Example 13–2: Evaluate the integral $\displaystyle\int e^{3x^2} x\,dx$.

Solution: $u = 3x^2 \quad \Rightarrow \quad du = 6x\,dx$

$$\Rightarrow \quad x\,dx = \frac{1}{6}\,du$$

$$\begin{aligned}\int e^{3x^2} x\,dx &= \int e^u \frac{1}{6}\,du \\ &= \frac{1}{6}e^u + c \\ &= \frac{e^{3x^2}}{6} + c.\end{aligned}$$

Example 13–3: Evaluate the integral $\displaystyle\int \left(x^3+6x^2\right)^7 \left(x^2+4x\right) dx$.

Solution: The substitution $u = x^3 + 6x^2$ gives:

$$\begin{aligned} du &= \left(3x^2 + 12x\right) dx \\ \frac{1}{3}\,du &= \left(x^2+4x\right) dx \end{aligned}$$

Rewriting the integral in terms of u, we obtain:

$$\begin{aligned}&\int \left(x^3+6x^2\right)^7 \left(x^2+4x\right) dx \\ &= \frac{1}{3}\int u^7\,du \\ &= \frac{u^8}{24} + c \\ &= \frac{\left(x^3+6x^2\right)^8}{24} + c\end{aligned}$$

Substitution in Definite Integrals: If u' is continuous on the interval $\left[a,\,b\right]$ and f is continuous on the range of u then:

$$\int_a^b f\big(u(x)\big)\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

Don't forget to transform the limits!

Example 13–4: Evaluate the integral $\displaystyle\int_1^2 \frac{x+2}{x^2+4x+1}\,dx$.

Solution: Use the substitution

$$\begin{aligned} u = x^2+4x+1 \quad &\Rightarrow \quad du = \left(2x+4\right) dx \\ &\Rightarrow \quad \frac{1}{2}\,du = \left(x+2\right) dx \end{aligned}$$

The new integral limits are:

$x = 1 \quad \Rightarrow \quad u = 6.$

$x = 2 \quad \Rightarrow \quad u = 13.$

Rewriting the integral in terms of u, we obtain:

$$\begin{aligned}\int_1^2 \frac{x+2}{x^2+4x+1}\,dx &= \int_6^{13} \frac{\frac{1}{2}du}{u} \\ &= \frac{1}{2}\ln|u|\,\Big|_6^{13} \\ &= \frac{1}{2}\ln 13 - \frac{1}{2}\ln 6 \\ &= \frac{1}{2}\ln\frac{13}{6}.\end{aligned}$$

Example 13–5: Evaluate the definite integral $\int_0^1 8x\left(x^2+2\right)^3 dx$.

Solution:

- Using $u = x^2+2, \quad du = 2x\,dx$ and

  $x = 0 \quad \Rightarrow \quad u = 2,$

  $x = 1 \quad \Rightarrow \quad u = 3.$

  we obtain:

$$\begin{aligned} I &= \int_2^3 4u^3\,du \\ &= u^4 \Big|_2^3 \\ &= 81-16 \\ &= 65. \end{aligned}$$

- Another idea is to evaluate it as an indefinite integral, rewrite u in terms of x and then use limits for x.
  Once again, using $u = x^2+2, \quad du = 2x\,dx$

$$\begin{aligned} \int 8x\left(x^2+2\right)^3 dx &= \int 4u^3\,du \\ &= u^4 + c \\ &= \left(x^2+2\right)^4 + c \end{aligned}$$

$$\begin{aligned} \int_0^1 8x\left(x^2+2\right)^3 dx &= \left(x^2+2\right)^4 \Big|_0^1 \\ &= 81-16 \\ &= 65. \end{aligned}$$

Example 13–6: Evaluate $\int_{-4}^4 \frac{x}{\sqrt{5-x}}\,dx$.

Solution: Using $u = 5-x,\ du = -dx,\ x = 5-u$ and

$x = -4 \quad \Rightarrow \quad u = 9, \qquad x = 4 \quad \Rightarrow \quad u = 1,$

we obtain:
$$\begin{aligned} I &= -\int_9^1 \frac{5-u}{\sqrt{u}}\,du \\ &= 10\,u^{\frac{1}{2}} - \frac{2}{3}\,u^{\frac{3}{2}} \Big|_1^9 \\ &= (30-18) - \left(10 - \frac{2}{3}\right) \\ &= \frac{8}{3}. \end{aligned}$$

Example 13–7: Evaluate $\int_0^2 8e^{x^4/5}x^3\,dx$.

Solution: Substitution: $u = \dfrac{x^4}{5}, \quad du = \dfrac{4}{5}\,x^3 dx.$

New limits:

$$x = 0 \quad \Rightarrow \quad u = 0, \quad x = 2 \quad \Rightarrow \quad u = \frac{16}{5}$$

$$\begin{aligned} I &= 10\int_0^{16/5} e^u\,du \\ &= 10e^u \Big|_0^{16/5} \\ &= 10\left(e^{16/5} - e^0\right) \\ &= 10\left(e^{16/5} - 1\right). \end{aligned}$$

EXERCISES

Evaluate the following integrals: (Hint: Use substitution.)

13–1) $\int (1+x)^3\,dx$

13–2) $\int (x^2+1)^4\,2x\,dx$

13–3) $\int e^{8t}\,dt$

13–4) $\int x^3 e^{-x^4}\,dx$

13–5) $\int \frac{1}{x+4}\,dx$

13–6) $\int \frac{3y}{(y^2-2)^4}\,dy$

13–7) $\int \frac{2x^3+3x}{x^4+3x^2+1}\,dx$

13–8) $\int e^{5x^3}\,7x^2\,dx$

13–9) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

13–10) $\int x(3x^2+7)^5\,dx$

Evaluate the following integrals: (Hint: Use substitution.)

13–11) $\int \sqrt{7x-12}\,dx$

13–12) $\int \frac{1}{1+5z}\,dz$

13–13) $\int \frac{3}{(2-x)^2}\,dx$

13–14) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

13–15) $\int 2y\sqrt{5-2y^2}\,dy$

13–16) $\int \frac{e^{1/x}}{x^2}\,dx$

13–17) $\int \frac{e^x+3x^2}{e^x+x^3+1}\,dx$

13–18) $\int \frac{6z^2+8z+3}{z+1}\,dz$

13–19) $\int \frac{\ln x}{x}\,dx$

13–20) $\int \frac{1}{x\,(2+\ln x)^3}\,dx$

Evaluate the following definite integrals:

13–21) $\displaystyle\int_0^1 (x^2+1)^3 x\,dx$

13–22) $\displaystyle\int_2^3 2e^{x^2} x\,dx$

13–23) $\displaystyle\int_0^1 x^3\sqrt[3]{x^4+1}\,dx$

13–24) $\displaystyle\int_0^1 (1+t)^3\,dt$

13–25) $\displaystyle\int_1^2 \frac{\ln(t^2)}{t}\,dt$

13–26) $\displaystyle\int_0^1 \frac{x+2x^3}{1+x^2+x^4}\,dx$

13–27) $\displaystyle\int_0^1 e^{x^2+x}(2x+1)\,dx$

13–28) $\displaystyle\int_0^{\ln 2} \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

13–29) $\displaystyle\int_0^{1/\sqrt{3}} (1+3t^2)^7 t\,dt$

13–30) $\displaystyle\int_1^{e^5} \frac{1+2\ln x}{x}\,dx$

ANSWERS

13–1) $\dfrac{(1+x)^4}{4}+c$

13–2) $\dfrac{(x^2+1)^5}{5}+c$

13–3) $\dfrac{e^{8t}}{8}+c$

13–4) $-\dfrac{e^{-x^4}}{4}+c$

13–5) $\ln|x+4|+c$

13–6) $-\dfrac{1}{2(y^2-2)^3}+c$

13–7) $\dfrac{1}{2}\ln|x^4+3x^2+1|+c$

13–8) $\dfrac{7}{15}e^{5x^3}+c$

13–9) $2e^{\sqrt{x}}+c$

13–10) $\dfrac{(3x^2+7)^6}{36}+c$

13–11) $\dfrac{2}{21}(7x-12)^{3/2}+c$

13–12) $\dfrac{1}{5}\ln|1+5z|+c$

13–13) $\dfrac{3}{2-x}+c$

13–14) $\dfrac{-2}{1+\sqrt{x}}+c$

13–15) $-\dfrac{1}{3}(5-2y^2)^{3/2}+c$

13–16) $-e^{1/x}+c$

13–17) $\ln|e^x+x^3+1|+c$

13–18) $3z^2+2z+\ln|z+1|+c$

13–19) $\dfrac{(\ln x)^2}{2}+c$

13–20) $\dfrac{-1}{2(2+\ln x)^2}+c$

13–21) $\dfrac{15}{8}$

13–22) e^9-e^4

13–23) $\dfrac{3}{16}\left(2^{4/3}-1\right)$

13–24) $\dfrac{15}{4}$

13–25) $(\ln 2)^2$

13–26) $\dfrac{\ln 3}{2}$

13–27) e^2-1

13–28) $\ln\left(\dfrac{5}{4}\right)$

13–29) $\dfrac{85}{16}$

13–30) 30

Chapter 14

Area Between Curves

Let f and g be continuous functions with $f(x) \geqslant g(x)$ on $\big[a,\, b\big]$. Then the area of the region bounded by the curves

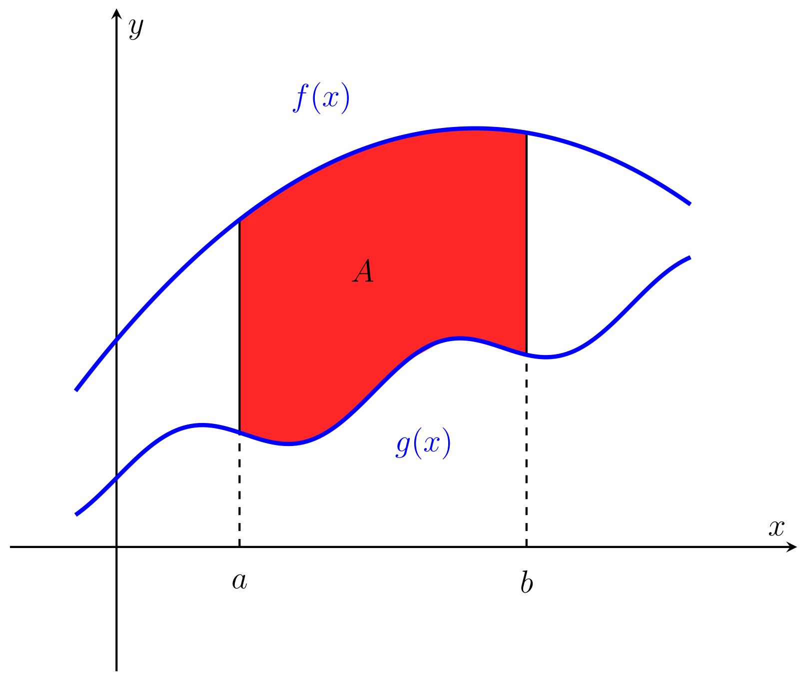
$$y = f(x), \quad y = g(x)$$

and the vertical lines

$$x = a, \quad x = b$$

is given by:

$$A = \int_a^b \Big(f(x) - g(x)\Big)\, dx$$



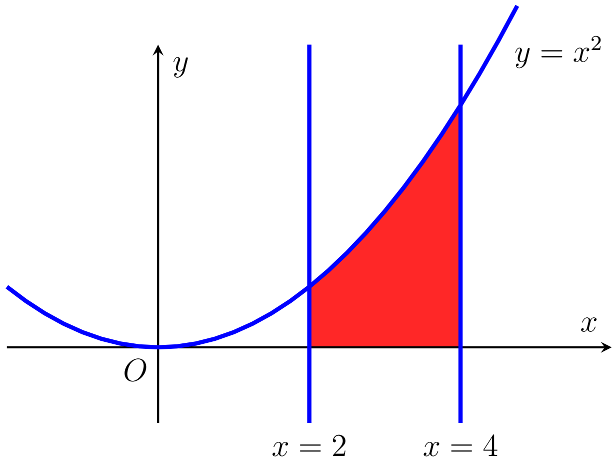
Note that the area must be positive, in other words we should integrate:

$$\text{TOP function} - \text{BOTTOM function}$$

Sometimes the curves intersect. In that case we have to find points of intersection.

Example 14–1: Find the area between the curve $y = x^2$, the lines
$x = 2$, $x = 4$ and $x-$axis.

Solution: Let's sketch the graph first:

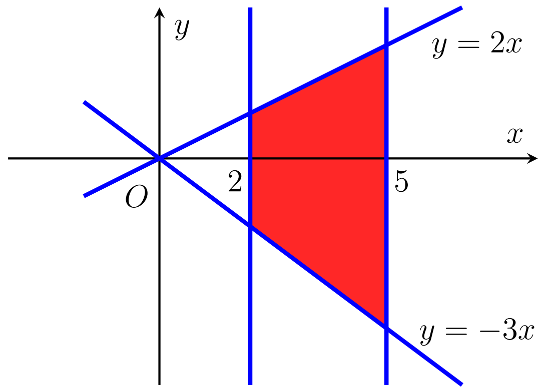


The integral that gives the shaded area is:

$$
\begin{aligned}
A &= \int_2^4 \left(x^2 - 0\right) dx \\
&= \left.\frac{x^3}{3}\right|_2^4 \\
&= \frac{4^3}{3} - \frac{2^3}{3} \\
&= \frac{56}{3}
\end{aligned}
$$

Example 14–2: Find the area bounded by the lines $y = 2x$, $y = -3x$, $x = 2$ and $x = 5$.

Solution: The lines $y = 2x$ and $y = -3x$ intersect at origin.

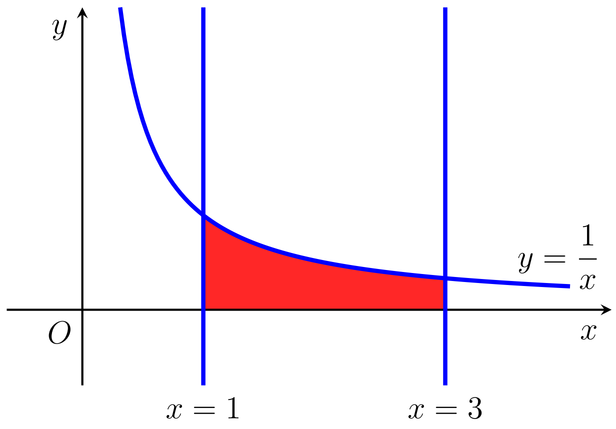


The area is:

$$
\begin{aligned}
A &= \int_2^5 \left(2x - (-3x)\right) dx \\
&= \int_2^5 5x\, dx \\
&= \left.\frac{5x^2}{2}\right|_2^5 \\
&= \frac{125}{2} - \frac{20}{2} \\
&= \frac{105}{2}
\end{aligned}
$$

Example 14–3: Find the area between the curve $y = \frac{1}{x}$, the lines
$x = 1$, $x = 3$ and $x-$axis.

Solution: The graph is:

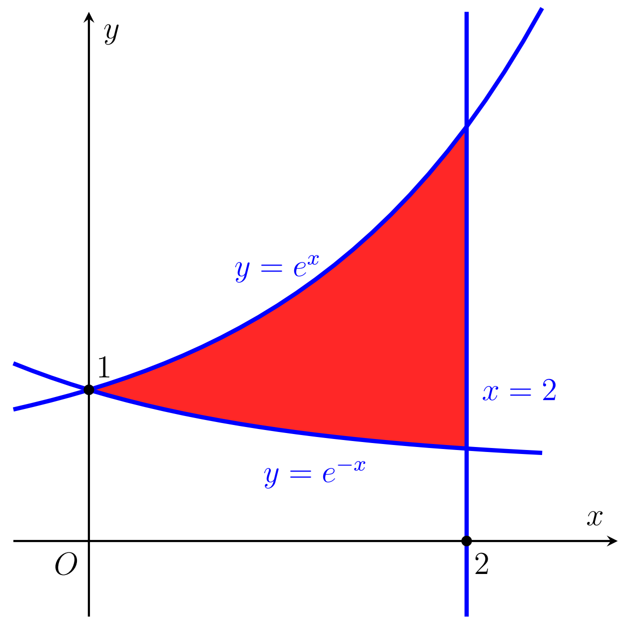


The integral that gives the shaded area is:

$$\begin{aligned} A &= \int_1^3 \left(\frac{1}{x} - 0\right) dx \\ &= \ln x \bigg|_1^3 \\ &= \ln 3 - \ln 1 \\ &= \ln 3 \end{aligned}$$

Example 14–4: Find the area between the curves $y = e^x$, e^{-x} and the lines $x = 2$.

Solution:

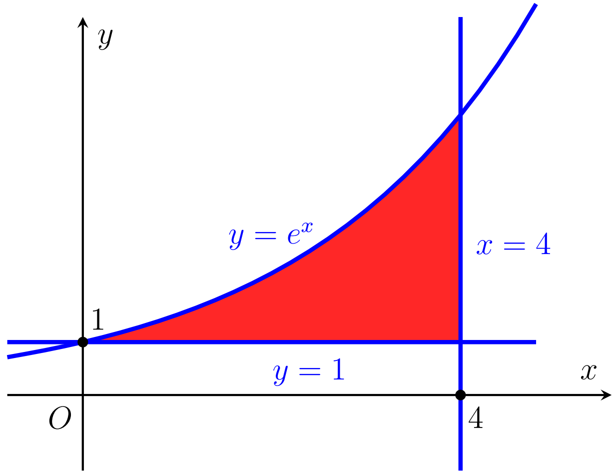


The area is:

$$\begin{aligned} A &= \int_0^2 \left(e^x - e^{-x}\right) dx \\ &= e^x + e^{-x} \bigg|_0^2 \\ &= \left(e^2 + e^{-2}\right) - \left(e^0 + e^0\right) \\ &= e^2 + e^{-2} - 2 \end{aligned}$$

Example 14–5: Find the area between the curve $y = e^x$, the lines
$y = 1$ and $x = 4$.

Solution: The region is between a curve and two lines. One is horizontal, the other is vertical.

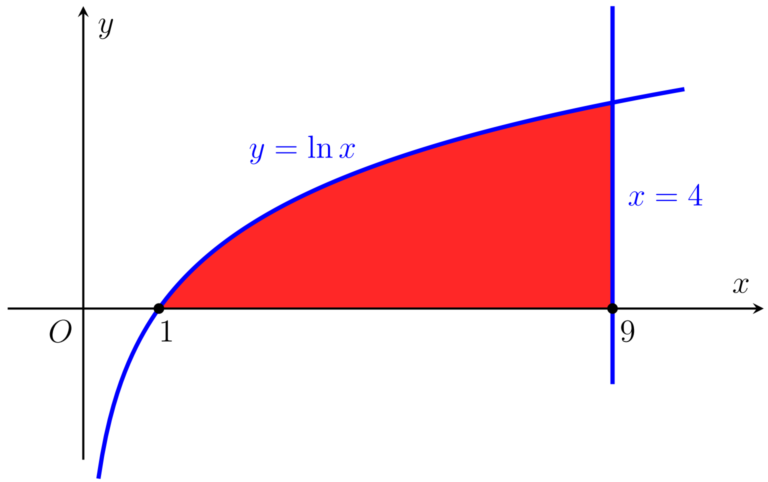


The area is:

$$
\begin{aligned}
A &= \int_0^4 (e^x - 1)\,dx \\
&= e^x - x \Big|_0^4 \\
&= \left(e^4 - 4\right) - \left(e^0 - 0\right) \\
&= e^4 - 5
\end{aligned}
$$

Example 14–6: Find the area between the curve $y = \ln x$, the line $x = 9$ and $x-$axis.

Solution: Remember that $\ln 1 = 0$ and $\ln 0$ is undefined.



The integral $\int \ln x\,dx$ can be evaluated using integration by parts, where $u = \ln x$ and $dv = dx$. The result is

$$\int \ln x\,dx = x\ln x - x + c$$

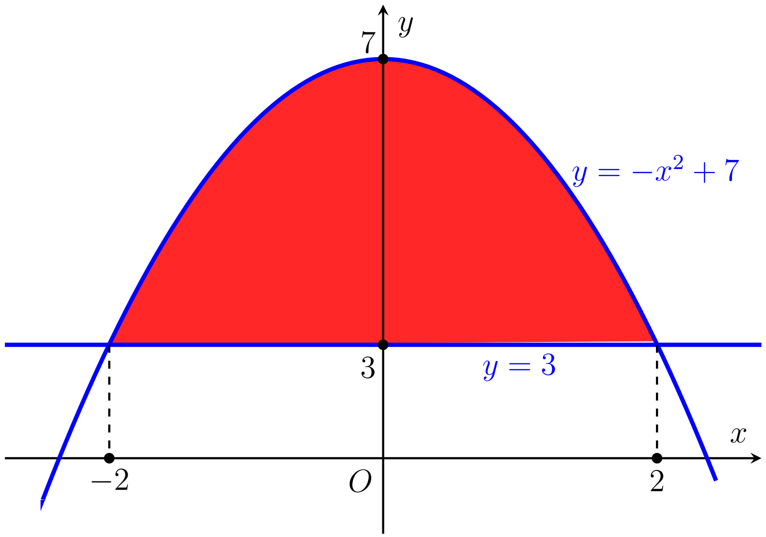
The area is:

$$
\begin{aligned}
A &= \int_1^9 \ln x\,dx \\
&= x\ln x - x \Big|_1^9 \\
&= \left(9\ln 9 - 9\right) - \left(1\ln 1 - 1\right) \\
&= 9\ln 9 - 8
\end{aligned}
$$

Example 14–7: Find the area between the curve $y = -x^2 + 7$ and the line $y = 3$.

Solution: First, we have to find intersection points.

$$-x^2 + 7 = 3 \quad \Rightarrow \quad x = \pm 2$$



The area is:

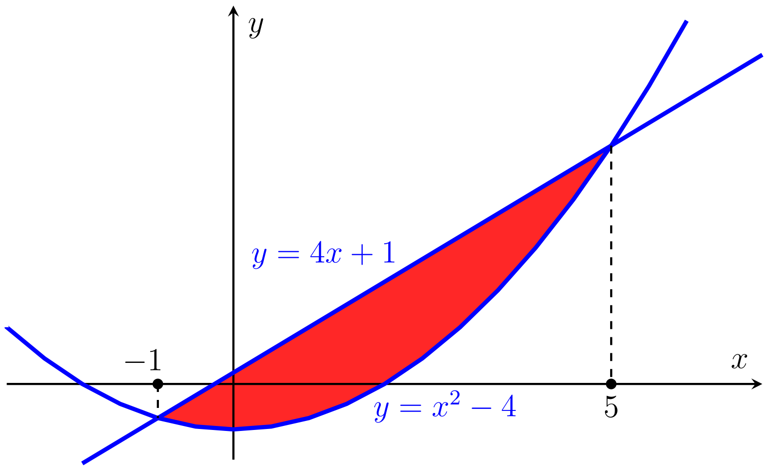
$$\begin{aligned} A &= \int_{-2}^{2} \big(-x^2 + 7 - 3\big)dx \\ &= -\frac{x^3}{3} + 4x \Big|_{-2}^{2} \\ &= \left(-\frac{8}{3} + 8\right) - \left(\frac{8}{3} - 8\right) \\ &= \frac{32}{3} \end{aligned}$$

Example 14–8: Find the area between the curve $y = x^2 - 4$ and the line $y = 4x + 1$.

Solution: First, we have to find intersection points.

$$x^2 - 4 = 4x + 1 \quad \Rightarrow \quad x^2 - 4x - 5 = 0$$

$$(x+1)(x-5) = 0 \quad \Rightarrow \quad x = 5 \quad \text{or} \quad x = -1$$



The area is:

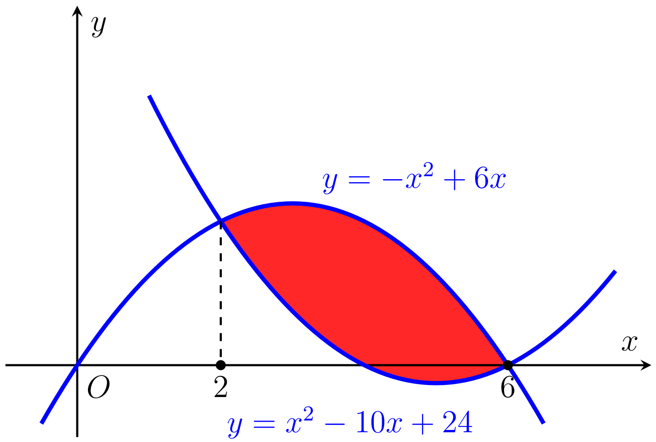
$$\begin{aligned} A &= \int_{-1}^{5} \big(4x + 1 - x^2 + 4\big)dx \\ &= -\frac{x^3}{3} + 2x^2 + 5x \Big|_{-1}^{5} \\ &= \left(-\frac{125}{3} + 50 + 25\right) - \left(\frac{1}{3} + 2 - 5\right) \\ &= 36 \end{aligned}$$

Example 14–9: Find the area between the curves $y = x^2 - 10x + 24$ and $y = -x^2 + 6x$.

Solution: First, we have to find intersection points.

$$x^2 - 10x + 24 = -x^2 + 6x \quad \Rightarrow \quad 2x^2 - 16x + 24 = 0$$

$$2(x-2)(x-6) = 0 \quad \Rightarrow \quad x = 2 \quad \text{or} \quad x = 6$$



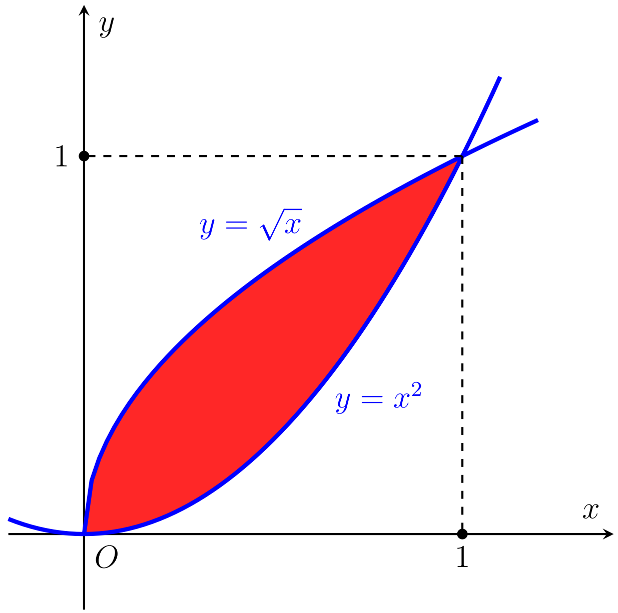
The area is:

$$\begin{aligned} A &= \int_2^6 \Big[\big(-x^2+6x\big) - \big(x^2-10x+24\big)\Big]dx \\ &= \int_2^6 \Big[-2x^2+16x-24\Big]dx \\ &= -\frac{2x^3}{3} + 8x^2 - 24x \Big|_2^6 \\ &= (-144+288+144) - \left(-\frac{16}{3}+32-48\right) \\ &= \frac{64}{3} \end{aligned}$$

Example 14–10: Find the area between the curves $y = x^2$ and $y = \sqrt{x}$.

Solution: First, we have to find intersection points.

$$x^2 = \sqrt{x} \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = 1$$



The area is:

$$\begin{aligned} A &= \int_0^1 \big(\sqrt{x} - x^2\big)dx \\ &= \frac{2}{3}x^{3/2} - \frac{x^3}{3}\Big|_0^1 \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \frac{1}{3} \end{aligned}$$

EXERCISES

Find the area of the region bounded by the given curves and lines:

14–1) $y = 3x^2 + 2x,\ x = 1,\ x = 3,\ x -$ axis.

14–2) $y = -x^2 + 8x,\ x = 0,\ x = 4,\ x -$ axis.

14–3) $y = 12x^3,\ x = 1,\ x = 2,\ x -$ axis.

14–4) $y = 5 - x^4,\ x = 0,\ x = 1,\ x -$ axis.

14–5) $y = \dfrac{3}{x^2},\ x = 1,\ x = 3,\ x -$ axis.

14–6) $y = \dfrac{1}{x},\ x = 2,\ x = 10,\ x -$ axis.

14–7) $y = \dfrac{8}{(x+1)^2},\ x = 0,\ x = 3,\ x -$ axis.

14–8) $y = e^x,\ x = 2,\ x = 5,\ x -$ axis.

14–9) $y = e^{-x},\ x = 0,\ x = 3,\ x -$ axis.

14–10) $y = e^{2x},\ x = -2,\ x = 1,\ x -$ axis.

14–11) $y = 3\sqrt{x},\ x = 1,\ x = 9,\ x -$ axis.

14–12) $y = \sqrt{x},\ x = 0,\ x = 6,\ x -$ axis.

14–13) $y = \ln x,\ x = 2,\ x = 4,\ x -$ axis.

Find the area of the region bounded by the given curves and lines: (Hint: First, find the intersection points.)

14–14) $y = x^2 + 1$ and $y = 2x + 1$.

14–15) $y = -x^2 + 5x$ and $y = 6$.

14–16) $y = x^2 - 6$ and $y = x$.

14–17) $y = -x^2 + 25$ and $y = -2x + 10$.

14–18) $y = 12x^2 - 3$ and $x -$ axis.

14–19) $y = 6x - 2x^2$ and $x -$ axis.

14–20) $y = x^2 - 2x - 6$ and $y = -x^2 + 2x$.

14–21) $y = x^2$ and $y = 6 - 5x^2$.

14–22) $y = e^x,\ y = e^{-x}$ and $x = 1$.

14–23) $y = e^{-x},\ y = 1$ and $x = 1$.

14–24) $y = 12\sqrt{x}$ and $y = 3x$.

14–25) $y = \dfrac{5}{x}$ and $y = 6 - x$.

ANSWERS

14–1) 34

14–2) $\frac{128}{3}$

14–3) 45

14–4) $\frac{24}{5}$

14–5) 2

14–6) $\ln 5$

14–7) 6

14–8) $e^5 - e^2$

14–9) $1 - \frac{1}{e^3}$

14–10) $\frac{e^2 - e^{-4}}{2}$

14–11) 52

14–12) $4\sqrt{6}$

14–13) $6\ln 2 - 2$

14–14) $\frac{4}{3}$

14–15) $\frac{1}{6}$

14–16) $\frac{125}{6}$

14–17) $\frac{256}{3}$

14–18) 2

14–19) 9

14–20) $\frac{64}{3}$

14–21) 8

14–22) $e + \frac{1}{e} - 2$

14–23) $\frac{1}{e}$

14–24) 128

14–25) $12 - 5\ln 5$

Chapter 15

Integration Techniques

Integration by Parts: Let's remember the product rule for derivatives:

$$[fg]' = f'g + fg'$$

Integrate both sides to obtain:

$$fg = \int f'g\,dx + \int fg'\,dx$$

Rearranging, we obtain:

$$\int fg'\,dx = fg - \int f'g\,dx$$

or in simpler notation:

$$\boxed{\int udv = uv - \int vdu}$$

Example 15–1: Evaluate $\displaystyle\int xe^x\,dx$.

Solution: Here, $udv = xe^x\,dx$. Let's choose u and dv as follows:

$$u = x \qquad \Rightarrow \qquad du = dx$$

$$dv = e^x\,dx \qquad \Rightarrow \qquad v = e^x$$

Using the integration by parts formula, we obtain:

$$\begin{aligned}\int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + c.\end{aligned}$$

Note that if we start with the alternative choice

$$u = e^x \qquad \Rightarrow \qquad du = e^x dx$$

$$dv = x\,dx \qquad \Rightarrow \qquad v = \frac{x^2}{2}$$

we obtain

$$\int xe^x\,dx = \frac{x^2}{2}\,e^x - \int \frac{x^2}{2}\,e^x\,dx$$

which correct but not helpful. The second integral is more complicated than the given integral.

Example 15–2: Evaluate the integral

$$\int (x+1)e^{3x}\,dx$$

Solution: We have to use integration by parts.

$$(x+1)e^{3x}\,dx = udv$$

$$\begin{aligned} u &= x+1 &\Rightarrow\quad du &= dx \\ dv &= e^{3x}\,dx &\Rightarrow\quad v &= \frac{e^{3x}}{3} \end{aligned}$$

$$\begin{aligned} \int (x+1)e^{3x}dx &= (x+1)\,\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}\,dx \\ &= \frac{1}{3}(x+1)e^{3x} - \frac{1}{3}\int e^{3x}\,dx \\ &= \frac{1}{3}(x+1)e^{3x} - \frac{e^{3x}}{9} + c \\ &= \frac{1}{3}\,xe^{3x} + \frac{2}{9}\,e^{3x} + c \end{aligned}$$

Example 15–3: Evaluate the integral

$$\int x^3 \ln x\,dx$$

Solution: We have to use integration by parts.

$$\begin{aligned} u &= \ln x &\Rightarrow\quad du &= \frac{dx}{x} \\ dv &= x^3\,dx &\Rightarrow\quad v &= \frac{x^4}{4} \end{aligned}$$

$$\begin{aligned} \int x^3 \ln x\,dx &= \frac{x^4}{4}\,\ln x - \int \frac{x^4}{4}\,\frac{dx}{x} \\ &= \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^3\,dx \\ &= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + c. \end{aligned}$$

Example 15–4: Evaluate $\int \ln x\,dx$.

Solution: $u = \ln x \quad \Rightarrow \quad du = \dfrac{dx}{x}$

$dv = dx \quad \Rightarrow \quad v = x$

Using the integration by parts formula, we obtain:

$$\begin{aligned}
\int \ln x\,dx &= x\,\ln x - \int x\,\frac{dx}{x} \\
&= x\,\ln x - \int dx \\
&= x\,\ln x - x + c.
\end{aligned}$$

Example 15–5: Evaluate $\int \ln^2 x\,dx$.

Solution: $u = \ln^2 x \quad \Rightarrow \quad du = \dfrac{2\ln x}{x}\,dx$

$dv = dx \quad \Rightarrow \quad v = x$

Using the integration by parts formula, we obtain:

$$\begin{aligned}
\int \ln^2 x\,dx &= x\,\ln^2 x - \int x\,\frac{2\ln x}{x}\,dx \\
&= x\,\ln^2 x - 2\int \ln x\,dx \\
&= x\,\ln^2 x - 2(x\,\ln x - x) + c \\
&= x\,\ln^2 x - 2x\,\ln x + 2x + c
\end{aligned}$$

Definite Integrals using Integration by Parts: It is the same formula but we have integration limits.

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

Don't forget the limits for the uv term!

Example 15–6: Evaluate $\int_0^1 xe^{-x}\,dx$.

Solution: $u = x \quad \Rightarrow \quad du = dx$

$dv = e^{-x}\,dx \quad \Rightarrow \quad v = -e^{-x}$

Using the formula, we obtain:

$$\begin{aligned}
\int_0^1 xe^{-x}\,dx &= -xe^{-x}\Big|_0^1 - \int_0^1 (-e^{-x})\,dx \\
&= -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx \\
&= \Big(-xe^{-x} - e^{-x}\Big)\Big|_0^1 \\
&= \Big(-e^{-1} - e^{-1}\Big) + \Big(0 + 1\Big) \\
&= 1 - 2e^{-1}
\end{aligned}$$

Example 15–7: Evaluate $\displaystyle\int_1^e x\ln x\,dx$.

Solution: $u = \ln x \quad \Rightarrow \quad du = \dfrac{dx}{x}$

$$dv = x\,dx \quad \Rightarrow \quad v = \frac{x^2}{2}$$

Using the formula, we obtain:

$$\begin{aligned}
\int_1^e x\ln x\,dx &= \frac{x^2}{2}\ln x\bigg|_1^e - \int_1^e \frac{x^2}{2}\,\frac{dx}{x} \\
&= \frac{x^2}{2}\ln x\bigg|_1^e - \frac{1}{2}\int_1^e x\,dx \\
&= \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\bigg|_1^e \\
&= \left(\frac{e^2}{2}\ln e - \frac{e^2}{4}\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\right) \\
&= \frac{e^2+1}{4}
\end{aligned}$$

Example 15–8: Evaluate $\displaystyle\int_1^e \frac{\ln x}{\sqrt{x}}\,dx$.

Solution: $u = \ln x \quad \Rightarrow \quad du = \dfrac{dx}{x}$

$$dv = \frac{dx}{\sqrt{x}} \quad \Rightarrow \quad v = 2\sqrt{x}$$

Using the formula, we obtain:

$$\begin{aligned}
\int_1^e \frac{\ln x}{\sqrt{x}}\,dx &= 2\sqrt{x}\,\ln x\bigg|_1^e - \int_1^e 2\sqrt{x}\,\frac{dx}{x} \\
&= 2\sqrt{x}\,\ln x\bigg|_1^e - \int_1^e \frac{2}{\sqrt{x}}\,dx \\
&= \left(2\sqrt{x}\,\ln x - 4\sqrt{x}\right)\bigg|_1^e \\
&= \left(2\sqrt{e}\,\ln e - 4\sqrt{e}\right) - \left(2\sqrt{1}\,\ln 1 - 4\sqrt{1}\right) \\
&= 4 - 2\sqrt{e}.
\end{aligned}$$

Partial Fractions Expansion: Given an algebraic expression like

$$\frac{3}{x-2}+\frac{5}{x+4}$$

we can write it with a common denominator as:

$$\frac{3}{x-2}+\frac{5}{x+4}=\frac{8x+2}{(x-2)(x+4)}.$$

To evaluate the integrals like $\int \frac{8x+2}{(x-2)(x+4)}\,dx$ we have to reverse this process.

$$\frac{8x+2}{(x-2)(x+4)}=\frac{A}{x-2}+\frac{B}{x+4}$$

$$8x+2=A(x+4)+B(x-2)$$

$$x=2 \quad \Rightarrow \quad A=\frac{18}{6}=3$$

$$x=-4 \quad \Rightarrow \quad B=\frac{-30}{-6}=5$$

Now, the integral can easily be evaluated in terms of logarithms.

For a given rational function $\frac{P(x)}{Q(x)}$, keep in mind the following:

- If degree of $P(x)$ is greater than (or equal to) the degree of $Q(x)$, divide them using polynomial division.
- If $Q(x)$ contains a power of the type $(ax+b)^n$, include all the terms
$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$$
in the expansion.

Example 15–9: Evaluate $\int \frac{4x+4}{(x-3)(x-2)}\,dx$.

Solution:

$$\frac{4x+4}{(x-3)(x-2)}=\frac{A}{x-3}+\frac{B}{x-2}$$

$$4x+4=A(x-2)+B(x-3)$$

$$x=3 \quad \Rightarrow \quad A=16$$

$$x=2 \quad \Rightarrow \quad B=-12$$

$$\int\left(\frac{16}{x-3}-\frac{12}{x-2}\right)dx$$
$$=16\ln|x-3|-12\ln|x-2|+c$$

Example 15–10: Evaluate $\int \frac{10x+15}{x^2+5x}\,dx$.

Solution:

$$\frac{10x+15}{x^2+5x}=\frac{A}{x}+\frac{B}{x+5}$$

$$10x+15=A(x+5)+Bx$$

$$x=-5 \quad \Rightarrow \quad B=7$$

$$x=0 \quad \Rightarrow \quad A=3$$

$$\int\left(\frac{3}{x}+\frac{7}{x+5}\right)dx=3\ln|x|+7\ln|x+5|+c$$

Example 15–11: Evaluate

$$\int \frac{13x^2 - 65x + 40}{x^3 - 9x^2 + 20x}\, dx$$

Solution:

$$\begin{aligned} \frac{13x^2 - 65x + 40}{x^3 - 9x^2 + 20x} &= \frac{13x^2 - 65x + 40}{x(x-4)(x-5)} \\ &= \frac{A}{x} + \frac{B}{x-4} + \frac{C}{x-5} \end{aligned}$$

$$13x^2 - 65x + 40 = A(x-4)(x-5) + Bx(x-5) + Cx(x-4)$$

We can find A, B, C as follows:

$$x = 0 \quad \Rightarrow \quad A = \frac{40}{20} = 2$$

$$x = 4 \quad \Rightarrow \quad B = \frac{-12}{-4} = 3$$

$$x = 5 \quad \Rightarrow \quad C = \frac{40}{5} = 8$$

$$\int \left(\frac{2}{x} + \frac{3}{x-4} + \frac{8}{x-5} \right) dx$$

$$= 2 \ln|x| + 3 \ln|x-4| + 8 \ln|x-5| + c$$

Example 15–12: Evaluate

$$\int \frac{dx}{x^2 - a^2}$$

where $a \neq 0$.

Solution:

$$\begin{aligned} \frac{1}{x^2 - a^2} &= \frac{1}{(x-a)(x+a)} \\ &= \frac{A}{x-a} + \frac{B}{x+a} \end{aligned}$$

$$1 = A(x+a) + B(x-a)$$

We can find A and B as follows:

$$x = a \quad \Rightarrow \quad A = \frac{1}{2a}$$

$$x = -a \quad \Rightarrow \quad B = \frac{1}{-2a}$$

$$\begin{aligned} \int \frac{dx}{x^2 - a^2} &= \int \left(\frac{1}{2a} \frac{1}{x-a} - \frac{1}{2a} \frac{1}{x+a} \right) dx \\ &= \frac{1}{2a} \big(\ln|x-a| - \ln|x+a| \big) + c \end{aligned}$$

Example 15–13: Evaluate

$$\int \frac{10x^2 - 22x + 7}{x(x-1)^2}\, dx$$

Solution:

$$\frac{10x^2 - 22x + 7}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$10x^2 - 22x + 7 = A(x-1)^2 + Bx(x-1) + Cx$$

We can solve these equations as follows:

$$x = 1 \quad \Rightarrow \quad C = 10 - 22 + 7 = -5$$

$$x = 0 \quad \Rightarrow \quad A = 7$$

If we expand the parentheses, we see that

$$10x^2 - 22x + 7 = (A+B)x^2 + (-2A - B + C)x + A$$

$$A + B = 10 \quad \Rightarrow \quad B = 3$$

$$\int \left(\frac{7}{x} + \frac{3}{x-1} - \frac{5}{(x-1)^2} \right) dx$$

$$= 7\ln|x| + 3\ln|x-1| + \frac{5}{x-1} + c$$

Example 15–14: Evaluate

$$\int \frac{5x^3 - 12x^2 - 6x - 5}{x^2 - 2x - 3}\, dx$$

Solution: First, we have to make a polynomial division to obtain:

$$\begin{aligned} \frac{5x^3 - 12x^2 - 6x - 5}{x^2 - 2x - 3} &= 5x - 2 + \frac{5x - 11}{x^2 - 2x - 3} \\ &= 5x - 2 + \frac{5x - 11}{(x-3)(x+1)} \end{aligned}$$

$$\frac{5x - 11}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$5x - 11 = A(x+1) + B(x-3)$$

$$x = 3 \quad \Rightarrow \quad A = 1$$

$$x = -1 \quad \Rightarrow \quad B = 4$$

$$\int \left(5x - 2 + \frac{1}{x-3} + \frac{4}{x+1} \right) dx$$

$$= \frac{5}{2}x^2 - 2x + \ln|x-3| + 4\ln|x+1| + c$$

EXERCISES

Evaluate the following integrals.

15–1) $\int (1+x)\, e^{2x}\, dx$

15–2) $\int \sqrt{x}\, \ln x\, dx$

15–3) $\int x e^{ax}\, dx$

15–4) $\int x^2 \ln x\, dx$

15–5) $\int \frac{\ln x}{\sqrt{x}}\, dx$

15–6) $\int x^2 e^x\, dx$

15–7) $\int \ln^3 x\, dx$

15–8) $\int x^p\, \ln x\, dx$

Evaluate the following definite integrals.

15–9) $\int_0^2 x e^{-x}\, dx$

15–10) $\int_1^5 x e^{-3x}\, dx$

15–11) $\int_1^2 x \ln x\, dx$

15–12) $\int_1^4 x^3 \ln x\, dx$

15–13) $\int_1^4 \frac{\ln x}{x^3}\, dx$

15–14) $\int_2^5 \ln(4x)\, dx$

15–15) $\int_1^3 4x e^{2x}\, dx$

15–16) $\int_1^8 \frac{\ln x}{x^{1/3}}\, dx$

Evaluate the following integrals.

15–17) $\displaystyle\int \frac{60}{x^2+6x}\,dx$

15–18) $\displaystyle\int \frac{5x+2}{x^2-x}\,dx$

15–19) $\displaystyle\int \frac{6}{2x^2+3x}\,dx$

15–20) $\displaystyle\int \frac{1}{x^2+x-6}\,dx$

15–21) $\displaystyle\int \frac{40}{x^2-16}\,dx$

15–22) $\displaystyle\int \frac{24x}{4x^2-24x+27}\,dx$

15–23) $\displaystyle\int \frac{4x+16}{x^3-4x}\,dx$

15–24) $\displaystyle\int \frac{3x^2+30}{x^3-7x^2+10x}\,dx$

15–25) $\displaystyle\int \frac{80}{x^3-4x^2}\,dx$

15–26) $\displaystyle\int \frac{12}{(x+5)(x+1)^2}\,dx$

ANSWERS

15–1) $\displaystyle\frac{(1+2x)\,e^{2x}}{4}+c$

15–2) $\displaystyle\frac{2}{3}\,x^{3/2}\ln x-\frac{4}{9}x^{3/2}+c$

15–3) $\displaystyle\frac{xe^{ax}}{a}-\frac{e^{ax}}{a^2}+c$

15–4) $\displaystyle\frac{x^3}{3}\left(\ln x-\frac{1}{3}\right)+c$

15–5) $2\sqrt{x}\ln x-4\sqrt{x}+c$

15–6) $x^2e^x-2xe^x+2e^x+c$

15–7) $x\,\ln^3 x-3x\,\ln^2 x+6x\,\ln x-6x+c$

15–8) $\displaystyle\frac{x^{p+1}\,\ln x}{p+1}-\frac{x^{p+1}}{(p+1)^2}+c$

15–9) $1 - 3e^{-2}$

15–10) $\dfrac{4}{9}e^{-3} - \dfrac{16}{9}e^{-15}$

15–11) $2\ln 2 - \dfrac{3}{4}$

15–12) $128\ln 2 - \dfrac{255}{16}$

15–13) $\dfrac{15}{64} - \dfrac{\ln 2}{16}$

15–14) $3\ln 4 + 5\ln 5 - 2\ln 2 - 3$

15–15) $5e^{6} - e^{2}$

15–16) $18\ln 2 - \dfrac{27}{4}$

15–17) $10\ln|x| - 10\ln|x+6| + c$

15–18) $7\ln|x-1| - 2\ln|x| + c$

15–19) $2\ln|x| - 2\ln|2x+3| + c$

15–20) $\dfrac{1}{5}\ln|x-2| - \dfrac{1}{5}\ln|x+3| + c$

15–21) $5\ln|x-4| - 5\ln|x+4| + c$

15–22) $9\ln|2x-9| - 3\ln|2x-3| + c$

15–23) $3\ln|x-2| - 4\ln|x| + \ln|x+2| + c$

15–24) $7\ln|x-5| - 7\ln|x-2| + 3\ln|x| + c$

15–25) $\dfrac{20}{x} + 5\ln|x-4| - 5\ln|x| + c$

15–26) $-\dfrac{3}{x+1} - \dfrac{3}{4}\ln|x+1| + \dfrac{3}{4}\ln|x+5| + c$

Chapter 16

Partial Derivatives

The partial derivative of $f(x,y)$ with respect to x at the point (a,b) is defined as:

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

We use several different notations to denote partial derivatives:

$$f_x, \quad \frac{\partial f}{\partial x}, \quad \left.\frac{\partial f}{\partial x}\right|_{(a,b)} \quad \text{etc.}$$

Second derivatives for $z = z(x,y)$ are:

$$z_{xx} = \frac{\partial^2 z}{\partial x^2}, \qquad z_{xy} = \frac{\partial^2 z}{\partial y\,\partial x}$$

Third and higher order derivatives are denoted similarly.

When we evaluate the partial derivative of a function with respect to a certain variable, we keep all the other variables fixed!

Example 16–1: Find f_x and f_y for $f(x,y) = x^5 + 7x^2y^3 + 4y^2$.

Solution: To find derivative with respect to x, we treat y as a constant and vice versa:

$$f_x = 5x^4 + 14xy^3$$

$$f_y = 21x^2y^2 + 8y$$

Example 16–2: Find f_{xy} and f_{yx} where

$$f(x,y) = x\ln y + 4xy - ye^x$$

Solution: $f_x = \ln y + 4y - ye^x$

$$\Rightarrow \quad f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{1}{y} + 4 - e^x$$

$$f_y = \frac{x}{y} + 4x - e^x$$

$$\Rightarrow \quad f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{1}{y} + 4 - e^x$$

Theorem: If the functions f_{xy} and f_{yx} are continuous in a disk containing (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$

Chain Rule: Suppose z is a differentiable function of the variables $x_1, x_2, \ldots, x_n$ and each x_i is a differentiable function of the variables $u_1, u_2, \ldots, u_m$. Then,

$$\frac{\partial z}{\partial u_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial u_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial u_i} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial u_i}$$

For example, if $z = z(x,y)$ and $x = x(t),\ y = y(t)$ then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Another example is:

$$f = f(x,y,z),\ x = x(s,t),\ y = y(s,t),\ z = z(s,t)$$

$$\Rightarrow \quad \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

Example 16–3: Let $f = xy + x^2 z$ where $x = t^2,\ y = e^t$ and $z = e^{2t}$. Find $\dfrac{df}{dt}$.

Solution:
$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= (y + 2xz)2t + xe^t + x^2 2e^{2t} \\ &= (e^t + 2t^2 e^{2t})2t + t^2 e^t + 2t^4 e^{2t} \\ &= (2t + t^2)e^t + (4t^3 + 2t^4)e^{2t}\end{aligned}$$

We can also start from $f = t^2 e^t + t^4 e^{2t}$.

Example 16–4: Let $z = x^2 + y^2 + 8xy \quad x = 2v + 4w,\ y = 5vw$. Find $\dfrac{\partial z}{\partial v}$.

Solution:
$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= (2x + 8y)2 + (2y + 8x)5w \\ &= (4v + 8w + 40vw)2 + (10vw + 16v + 32w)5w \\ &= 8v + 16w + 160vw + 50vw^2 + 160w^2\end{aligned}$$

Alternatively, we can insert the values of x and y in the beginning:

$$z = (2v + 4w)^2 + (5vw)^2 + 8(2v + 4w)(5vw)$$

$$z = 4v^2 + 16vw + 16w^2 + 25v^2w^2 + 80v^2w + 160vw^2$$

$$\frac{\partial z}{\partial v} = 8v + 16w + 50vw^2 + 160vw + 160w^2$$

Implicit Differentiation: If $F(x, y) = const.$ is a function that defines y in terms of x implicitly, then

$$\frac{dF}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example 16–5: Using the equation $x^2 + xy^4 + z^2 = 12$, find $\dfrac{\partial z}{\partial y}$.

Solution: Here, $F = x^2 + xy^4 + z^2$, so:

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4xy^3}{2z}$$

In this example, it is possible to solve for z explicitly:

$$z = \sqrt{12 - x^2 - xy^4}$$

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{1}{2}\left(12 - x^2 - xy^4\right)^{-1/2}(-4xy^3) \\
&= -\frac{4xy^3}{2\left(12 - x^2 - xy^4\right)^{1/2}} \\
&= -\frac{4xy^3}{2z}
\end{aligned}$$

Example 16–6: Find z_x at $(3, 4, 1)$ using the equation

$$xy + \frac{2x}{z} - y\ln z = 18$$

Solution: Derivative of both sides with respect to x gives:

$$y + \frac{2}{z} - \frac{2x}{z^2}z_x - \frac{y}{z}z_x = 0$$

$$\Rightarrow \quad z_x = \frac{y + \dfrac{2}{z}}{\dfrac{2x}{z^2} + \dfrac{y}{z}}$$

$$z_x(3, 4, 1) = \frac{4 + 2}{6 + 4} = \frac{3}{5}$$

Example 16–7: Find $z_y(4, 2, 1)$ using the equation

$$xz^3 - \frac{3x}{y} + y\ln z = 10$$

Solution: Using the formula, we obtain:

$$\begin{aligned}
z_y &= -\frac{F_y}{F_z} \\
&= -\frac{\dfrac{3x}{y^2} + \ln z}{3xz^2 + \dfrac{y}{z}} \\
z_y(4, 2, 1) &= -\frac{3 + 0}{12 + 2} = -\frac{3}{14}
\end{aligned}$$

EXERCISES

Find f_x and f_y for the following functions:

16–1) $f(x, y) = 1 + 5x + 7xy^2 - 20y^4$

16–2) $f(x, y) = 4x^3y^5$

16–3) $f(x, y) = \dfrac{x^2}{y^4}$

16–4) $f(x, y) = x^2e^y + y$

16–5) $f(x, y) = x\ln x - yx^2 \ln y$

16–6) $f(x, y) = 4x + 5yx\sqrt{x} - \dfrac{3}{y}$

Find f_{xx}, f_{xy} and f_{yy} for the following functions:

16–7) $f(x, y) = 6x - 8xy - 2y^5$

16–8) $f(x, y) = 4x^3 + 7y^2$

16–9) $f(x, y) = y^3e^x$

16–10) $f(x, y) = \ln x + 5y$

16–11) $f(x, y) = \dfrac{x^5}{y^5}$

16–12) $f(x, y) = e^{3x} \ln\left(y^4\right)$

16–13) If $3xy + z^3y - 12xyz = 20$, find $\dfrac{\partial z}{\partial x}$.

16–14) If $zxy + y^4e^z = 3$, find $\dfrac{\partial z}{\partial y}$.

16–15) If $2xy^5 + xyz + ze^y = 12$ find $\dfrac{\partial y}{\partial z}$.

16–16) If $x^3y + y^4z + z^5w + w^6 = 6$ find $\dfrac{\partial y}{\partial w}$.

16–17) If $y\ln x + x^3\ln z + 2yz^3 + xyz = 18$ find $\dfrac{\partial x}{\partial z}$.

16–18) If $xy^2w^3 + yze^z + 3xy^3 - 5x^3w = 0$ find $\dfrac{\partial w}{\partial z}$.

16–19) If $f(x, y) = x^2y^3e^{-y}$, find $\dfrac{\partial f}{\partial y}$.

16–20) If $f(x, y) = e^{xy} + x\ln y$, find f_{xy}.

16–21) If $x^2z^2 + 2xy + y^3z^5 = 24$, find $\dfrac{\partial z}{\partial y}$.

16–22) If $\dfrac{x}{y} + \dfrac{y}{z} = 1$, find $\dfrac{\partial z}{\partial x}$.

16–23) If $x^2 + y^4 + z^8 + w^{10} = 4$, find $\dfrac{\partial z}{\partial x}$ at the point $(1, 1, 1, 1)$.

ANSWERS

16–1) $f_x = 5 + 7y^2, \quad f_y = 14xy - 80y^3.$

16–2) $f_x = 12x^2y^5, \quad f_y = 20x^3y^4.$

16–3) $f_x = \dfrac{2x}{y^4}, \quad f_y = -\dfrac{4x^2}{y^5}.$

16–4) $f_x = 2xe^y, \quad f_y = x^2e^y + 1.$

16–5) $f_x = \ln x + 1 - 2yx\ln y, \quad f_y = -x^2\ln y - x^2.$

16–6) $f_x = 4 + \dfrac{15}{2}y\sqrt{x}, \quad f_y = 5x\sqrt{x} + \dfrac{3}{y^2}.$

16–7) $f_{xx} = 0, \quad f_{xy} = -8, \quad f_{yy} = -40y^3.$

16–8) $f_{xx} = 24x, \quad f_{xy} = 0, \quad f_{yy} = 14.$

16–9) $f_{xx} = y^3e^x, \quad f_{xy} = 3y^2e^x, \quad f_{yy} = 6ye^x.$

16–10) $f_{xx} = -\dfrac{1}{x^2}, \quad f_{xy} = 0, \quad f_{yy} = 0.$

16–11) $f_{xx} = \dfrac{20x^3}{y^5}, \quad f_{xy} = -\dfrac{25x^4}{y^6}, \quad f_{yy} = \dfrac{30x^5}{y^7}.$

16–12) $f_{xx} = 9e^{3x}\ln\left(y^4\right), \quad f_{xy} = \dfrac{12e^{3x}}{y}, \quad f_{yy} = -\dfrac{4e^{3x}}{y^2}.$

16–13) $\dfrac{\partial z}{\partial x} = -\dfrac{3y - 12yz}{3z^2y - 12xy}$

16–14) $\dfrac{\partial z}{\partial y} = -\dfrac{zx + 4y^3e^z}{xy + y^4e^z}$

16–15) $\dfrac{\partial y}{\partial z} = -\dfrac{xy + e^y}{10xy^4 + xz + ze^y}$

16–16) $\dfrac{\partial y}{\partial w} = -\dfrac{z^5 + 6w^5}{x^3 + 4y^3z}$

16–17) $\dfrac{\partial x}{\partial z} = -\dfrac{\frac{x^3}{z} + 6yz^2 + xy}{\frac{y}{x} + 3x^2\ln z + yz}$

16–18) $\dfrac{\partial w}{\partial z} = -\dfrac{ye^z + yze^z}{3xy^2w^2 - 5x^3}$

16–19) $x^2(3y^2 - y^3)e^{-y}$

16–20) $e^{xy} + xye^{xy} + \dfrac{1}{y}$

16–21) $-\dfrac{2x + 3y^2z^5}{2x^2z + 5y^3z^4}$

16–22) $\dfrac{z^2}{y^2}$

16–23) $-\dfrac{1}{4}$

Chapter 17

Local Extrema

Second Derivative Test for Local Extrema: Suppose that $f(x, y)$ and its first and second partial derivatives are continuous at and around the point (a, b) and also that

$$f_x(a, b) = f_y(a, b) = 0$$

Let's calculate the second derivatives at (a, b):

$$A = f_{xx}(a, b), \quad B = f_{yy}(a, b), \quad C = f_{xy}(a, b)$$

Δ (Delta) at (a, b) is defined as:

$$\Delta = AB - C^2$$

- If $\Delta > 0$ and $A > 0$ then f has a local minimum at (a, b).
- If $\Delta > 0$ and $A < 0$ then f has a local maximum at (a, b).
- If $\Delta < 0$ then f has a saddle point at (a, b).
- If $\Delta = 0$ then the test is inconclusive.

Example 17–1: Find the local extrema of

$$f(x, y) = x^2 + 2y^2 - 4y + 5$$

Solution: $f_x = 0 \quad \Rightarrow \quad 2x = 0$

$f_y = 0 \quad \Rightarrow \quad 4y - 4 = 0$

The only critical point is $(x, y) = (0, 1)$

$$f_{xx} = 2, \quad \Rightarrow \quad A = 2$$
$$f_{yy} = 4, \quad \Rightarrow \quad B = 4$$
$$f_{xy} = 0, \quad \Rightarrow \quad C = 0$$

$$\Rightarrow \quad \Delta = 2 \cdot 4 - 0 = 8$$

$\Delta > 0$, $A > 0 \quad \Rightarrow \quad (0, 1)$ is a local minimum.

Example 17–2: Find and classify the critical points of

$$f(x,y) = -x^2 - 4y^2 + 2xy + 12y + 15$$

Solution: The first derivatives are:

$$f_x = 0 \quad \Rightarrow \quad -2x + 2y = 0$$

$$f_y = 0 \quad \Rightarrow \quad -8y + 2x + 12 = 0$$

The solution of this system of equations is:

$$y = x \quad \Rightarrow \quad -8x + 2x + 12 = 0$$

$$\Rightarrow \quad 6x = 12$$

$$\Rightarrow \quad x = 2, \quad y = 2.$$

The only critical point is $(2,2)$

$$f_{xx} = -2 \quad \Rightarrow \quad A = -2$$

$$f_{yy} = -8 \quad \Rightarrow \quad B = -8$$

$$f_{xy} = 2 \quad \Rightarrow \quad C = 2$$

$$\Delta = (-2) \cdot (-8) - 2^2 = 12$$

$\Delta > 0, \quad A < 0 \quad \Rightarrow \quad (2,2)$ is a local maximum.

Example 17–3: Find and classify the critical points of

$$f(x,y) = x^2 - y^2 + 1$$

Solution: The first derivatives are:

$$f_x = 0 \quad \Rightarrow \quad 2x = 0$$

$$f_y = 0 \quad \Rightarrow \quad -2y = 0$$

The solution is $x = 0, \quad y = 0.$

$$f_{xx} = 2 \quad \Rightarrow \quad A = 2$$

$$f_{yy} = -2 \quad \Rightarrow \quad B = -2$$

$$f_{xy} = 0 \quad \Rightarrow \quad C = 0$$

$$\Delta = 2 \cdot (-2) - 0 = -4$$

$\Delta < 0 \quad \Rightarrow \quad (0,0)$ is a saddle point.

Example 17–4: Find and classify the critical points of

$$f(x,y) = 3xy - 12x + 5$$

Solution: $f_x = 0 \quad \Rightarrow \quad 3y - 12 = 0$

$$f_y = 0 \quad \Rightarrow \quad 3x = 0$$

The solution is $x = 0, \quad y = 4.$

$$f_{xx} = 0 \quad \Rightarrow \quad A = 0$$

$$f_{yy} = 0 \quad \Rightarrow \quad B = 0$$

$$f_{xy} = 3 \quad \Rightarrow \quad C = 3$$

$$\Delta = 0 - 3^2 = -9$$

$\Delta < 0 \quad \Rightarrow \quad (0,4)$ is a saddle point.

Example 17–5: Find and classify the critical points of

$$f(x,y) = 2x^3 - 3x^2 + 12xy - 2y^2$$

Solution: The first derivatives give:

$$f_x = 0 \quad \Rightarrow \quad 6x^2 - 6x + 12y = 0$$

$$f_y = 0 \quad \Rightarrow \quad 12x - 4y = 0$$

The second equation gives $y = 3x$. Using $y = 3x$ in the first equation we obtain:

$$6x^2 + 30x = 0 \quad \Rightarrow \quad x = 0, \quad x = -5$$

$$x = 0 \quad \Rightarrow \quad y = 0$$

$$x = -5 \quad \Rightarrow \quad y = -15$$

There are two critical points: $(0,0)$ and $(-5,-15)$.

$$f_{xx} = 12x - 6, \quad f_{yy} = -4, \quad f_{xy} = 12$$

- For $(0,0)$:

  $A = -6, \quad B = -4, \quad C = 12$

  $\Rightarrow \quad \Delta = -120$

  $\Delta < 0 \quad \Rightarrow \quad (0,0)$ is a saddle point.

- For $(-5,-15)$:

  $A = -66, \quad B = -4, \quad C = 12$

  $\Rightarrow \quad \Delta = 120$

  $\Delta > 0, \quad A < 0$

  $\Rightarrow \quad (-5,-15)$ is a local maximum.

Example 17–6: Find and classify the critical points of

$$f(x,y) = x^2y - 4xy + \frac{y^3}{48} + 3y$$

Solution: $f_x = 0 \quad \Rightarrow \quad 2xy - 4y = 0$

$$f_y = 0 \quad \Rightarrow \quad x^2 - 4x + \frac{y^2}{16} + 3 = 0$$

The first equation gives: $x = 2$ or $y = 0$.

$x = 2 \quad \Rightarrow \quad y^2 = 16 \quad \Rightarrow \quad y = \pm 4.$

$y = 0 \quad \Rightarrow \quad x^2 - 4x + 3 = 0 \quad \Rightarrow \quad x = 1$ or $x = 3$.

Critical points: $(2,4)$, $(2,-4)$, $(1,0)$ and $(3,0)$.

$$f_{xx} = 2y, \quad f_{yy} = \frac{y}{8}, \quad f_{xy} = 2x - 4$$

- For $(2,4)$:

  $A = 8, \quad B = \frac{1}{2}, \quad C = 0 \quad \Rightarrow \quad \Delta = 4$

  $\Delta > 0$ and $A > 0 \quad \Rightarrow$ local minimum.

- For $(2,-4)$:

  $A = -8, \ B = -\frac{1}{2}, \quad C = 0 \quad \Rightarrow \quad \Delta = 4$

  $\Delta > 0$ and $A < 0 \quad \Rightarrow$ local maximum.

- For $(1,0)$:

  $A = 0, \quad B = 0, \quad C = -2 \quad \Rightarrow \quad \Delta = -4$

  $\Delta < 0 \quad \Rightarrow$ saddle point.

- For $(3,0)$:

  $A = 0, \quad B = 0, \quad C = 2 \quad \Rightarrow \quad \Delta = -4$

  $\Delta < 0 \quad \Rightarrow$ saddle point.

EXERCISES

Find and classify all the critical points the following functions:

17–1) $f(x,y) = 6x^2 + y^2 + 15x - 7y + 6$

17–2) $f(x,y) = 4x^2 + 6xy + 3y^2 + 5x - y + 16$

17–3) $f(x,y) = x^2 + 2xy - 2y^2 + 3x + 6y + 8$

17–4) $f(x,y) = -4x^2 - y^2 + 12x + 9$

17–5) $f(x,y) = 2x^2 + 3xy + y^2 + 20x + 40y$

17–6) $f(x,y) = -x^2 + 4xy + 14y^2 + x + 2$

17–7) $f(x,y) = x^2 + 8xy + 18y^2 - 16y + 13$

17–8) $f(x,y) = 2x^2 + 6xy + 2y^2 + 44x + 16y$

17–9) $f(x,y) = -5x^2 + 5xy - 3y^2 + 14y + 8$

17–10) $f(x,y) = -3x^2 - 2xy - y^2 + 5x - 7y$

17–11) $f(x,y) = 4y(x-6) + (x-6)^2$

17–12) $f(x,y) = (x-1)^2 + (y+1)^2$

Find and classify all the critical points the following functions:

17–13) $f(x,y) = 4x^3 + 2xy + \dfrac{y^3}{2}$

17–14) $f(x,y) = 5x^3 - 3xy - 5y^3$

17–15) $f(x,y) = 2x^3 + 4y^3 - 9x^2 - 6y^2 + 24$

17–16) $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 + 6$

17–17) $f(x,y) = -25x^3 + 100x^2 + 20xy + 4y^2$

17–18) $f(x,y) = x^3 + 4xy + 27y^3$

17–19) $f(x,y) = 3x^3 + 18xy + 3y^3$

17–20) $f(x,y) = x^4 + 8y^2 - 32xy$

17–21) $f(x,y) = 4xy - x^4 - y^4$

17–22) $f(x,y) = x^2 - x^2y + 12y^3$

ANSWERS

17–1) $(x, y) = \left(-\frac{5}{4}, \frac{7}{2}\right)$ local minimum.

17–2) $(x, y) = \left(-3, \frac{19}{6}\right)$ local minimum.

17–3) $(x, y) = \left(-2, \frac{1}{2}\right)$ saddle point .

17–4) $(x, y) = \left(\frac{3}{2}, 0\right)$ local maximum .

17–5) $(x, y) = (-80, 100)$ saddle point .

17–6) $(x, y) = \left(\frac{7}{18}, -\frac{1}{18}\right)$ saddle point .

17–7) $(x, y) = (-16, 4)$ local minimum .

17–8) $(x, y) = (4, -10)$ saddle point .

17–9) $(x, y) = (2, 4)$ local maximum .

17–10) $(x, y) = \left(3, -\frac{13}{2}\right)$ local maximum .

17–11) $(x, y) = (6, 0)$ saddle point .

17–12) $(x, y) = (1, -1)$ local minimum .

17–13) $(x, y) = (0, 0)$ saddle point,
$(x, y) = \left(-\frac{1}{3}, -\frac{2}{3}\right)$ local maximum .

17–14) $(x, y) = (0, 0)$ saddle point,
$(x, y) = \left(-\frac{1}{5}, \frac{1}{5}\right)$ local minimum .

17–15) $(x, y) = (0, 0)$ local maximum,
$(x, y) = (3, 1)$ local minimum,
$(x, y) = (0, 1)$ and $(3, 0)$ saddle points.

17–16) $(x, y) = (-2, 0)$ local maximum,
$(x, y) = (0, 2)$ local minimum,
$(x, y) = (0, 0)$ and $(-2, 2)$ saddle points.

17–17) $(x,y) = (2,-5)$ saddle point,

$(x,y) = (0,0)$ local minimum .

17–18) $(x,y) = (0,0)$ saddle point,

$(x,y) = \left(-\frac{4}{9}, -\frac{4}{27}\right)$ local maximum .

17–19) $(x,y) = (0,0)$ saddle point,

$(x,y) = (-2,-2)$ local maximum .

17–20) $(x,y) = (4,8)$ and $(-4,-8)$ local minima,

$(x,y) = (0,0)$ saddle point.

17–21) $(x,y) = (1,1)$ and $(1,-1)$ local maxima,

$(x,y) = (0,0)$ saddle point.

17–22) At $(x,y) = (0,0)$ test fails.

$(x,y) = (6,1)$ saddle point.

Chapter 18

Lagrange Multipliers

Sometimes we need to find maximum or minimum of a function subject to certain constraints.

For example: Find the maximum of

$$f(x, y) = xy$$

subject to the constraint

$$2x + 5y = 20$$

In this equation, the variables x and y can not take on any values. They have to satisfy the constraint given by $2x + 5y = 20$. So we can not just evaluate derivatives and set them equal to zero. But we can eliminate y to obtain:

$$y = \frac{20 - 2x}{5}$$

$$\begin{aligned} f(x) &= x\left(4 - \frac{2}{5}x\right) \\ &= 4x - \frac{2}{5}x^2 \end{aligned}$$

$$f'(x) = 4 - \frac{4}{5}x = 0 \quad \Rightarrow \quad x = 5,\ y = 2$$

It is not always possible to eliminate variables in this way. Lagrange multipliers is an alternative method. We consider the equations:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\ g(x, y) &= \text{constant} \end{aligned}$$

This is a set of three equation in three unknowns x, y and λ. If the maximum and minimum of f exist, they are among the points in the solution set.
Note that this method won't work for the exceptional case

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$$

Example 18–1: Find critical points of the function

$$f(x,y) = 3xy$$

subject to the constraint $g(x,y) = 5x + 2y = 20$.

Solution: Using the equations

$$\begin{aligned} f_x &= \lambda\, g_x \\ f_y &= \lambda\, g_y \end{aligned}$$

and the constraint, we obtain three equations in three unknowns:

$$\begin{aligned} 3y &= 5\lambda \\ 3x &= 2\lambda \\ 5x + 2y &= 20 \end{aligned}$$

We can eliminate λ as follows:

$$\frac{3y}{5} = \lambda = \frac{3x}{2}$$

$$\Rightarrow \quad 2y = 5x$$

Now using the third equation we obtain:

$$\begin{aligned} 5x + 5x &= 20 \\ 10x &= 20 \\ \Rightarrow &\quad x = 2 \\ y = \frac{5}{2}x \quad \Rightarrow &\quad y = 5 \end{aligned}$$

The critical point is: $(2,5)$

Example 18–2: Find critical points of the function $f(x,y) = 4x^2 + y^2$ subject to the constraint $g(x,y) = 6x + 5y = 34$.

Solution: Using the equations

$$\begin{aligned} f_x &= \lambda\, g_x \\ f_y &= \lambda\, g_y \end{aligned}$$

and the constraint, we obtain:

$$\begin{aligned} 8x &= 6\lambda \\ 2y &= 5\lambda \\ 6x + 5y &= 34 \end{aligned}$$

We can eliminate λ as follows:

$$\frac{8x}{6} = \lambda = \frac{2y}{5}$$

$$\Rightarrow \quad 40x = 12y \quad \Rightarrow \quad 10x = 3y$$

Now using the third equation we obtain:

$$\begin{aligned} 6x + \frac{50x}{3} &= 34 \\ 68x &= 34 \cdot 3 \\ \Rightarrow &\quad x = \frac{3}{2} \\ y = \frac{10x}{3} \quad \Rightarrow &\quad y = 5 \end{aligned}$$

The critical point is:

$$\left(\frac{3}{2}, 5\right)$$

Example 18–3: Find critical points of the function

$$f(x,y) = x^3y^2$$

subject to the constraint

$$g(x,y) = 24x + y = 10$$

Solution: We obtain the set of equations:

$$\begin{aligned} 3x^2y^2 &= 24\lambda \\ 2x^3y &= \lambda \\ 24x + y &= 10 \end{aligned}$$

We can eliminate λ as follows:

$$\frac{3x^2y^2}{24} = \lambda = 2x^3y$$

If $x \neq 0$ and $y \neq 0$ we obtain

$$\Rightarrow \quad 3y = 48x \quad \Rightarrow \quad y = 16x$$

$$\begin{aligned} 24x + 16x &= 10 \\ 40x &= 10 \\ \Rightarrow \quad x &= \frac{1}{4} \end{aligned}$$

$$y = 16x \quad \Rightarrow \quad y = 4$$

We should also consider the cases $x = 0$ and $y = 0$ separately. The critical points are:

$$\left(\frac{1}{4}, 4\right), \quad (0, 10), \quad \left(\frac{5}{12}, 0\right)$$

Example 18–4: Find critical points of the function

$$f(x,y) = 4x + 8y$$

subject to the constraint

$$g(x,y) = x^2 + 5y^2 = 5$$

Solution: We obtain the set of equations:

$$\begin{aligned} 4 &= 2\lambda x \\ 8 &= 10\lambda y \\ x^2 + 5y^2 &= 5 \end{aligned}$$

We can eliminate λ as follows:

$$\frac{4}{2x} = \lambda = \frac{8}{10y}$$

$$\Rightarrow \quad 40y = 16x \quad \Rightarrow \quad y = \frac{2x}{5}$$

Replacing y in the equation $x^2 + 5y^2 = 5$, we obtain

$$\begin{aligned} x^2 + \frac{4x^2}{5} &= 5 \\ \frac{9x^2}{5} &= 5 \\ x^2 &= \frac{25}{9} \\ \Rightarrow \quad x &= \pm\frac{5}{3} \end{aligned}$$

$y = \frac{2x}{5}$, therefore the critical points are:

$$\left(\frac{5}{3}, \frac{2}{3}\right), \quad \left(-\frac{5}{3}, -\frac{2}{3}\right)$$

Example 18–5: Find critical points of the function

$$f(x, y) = x^2 - 6xy + y^2$$

subject to the constraint

$$g(x, y) = x + y = 6$$

Solution: We obtain the set of equations:

$$\begin{aligned} 2x - 6y &= \lambda \\ -6x + 2y &= \lambda \\ x + y &= 6 \end{aligned}$$

We can eliminate λ as follows:

$$2x - 6y = \lambda = -6x + 2y$$

$$\Rightarrow \quad 48x = 8y$$

$$\Rightarrow \quad x = y$$

$$x + y = 6$$

$$\Rightarrow \quad 2x = 6$$

$$\Rightarrow \quad x = 3, \quad y = 3$$

The critical point is: $(3, 3)$

Example 18–6: Find critical points of the function

$$f(x, y) = 36x + 2y$$

subject to the constraint

$$g(x, y) = 9x^2 + 2y = 48$$

Solution: We obtain the set of equations:

$$\begin{aligned} 36 &= 18x\lambda \\ 2 &= 2\lambda \\ 9x^2 + 2y &= 48 \end{aligned}$$

We can eliminate λ as follows:

$$\frac{36}{18x} = \lambda = \frac{2}{2} = 1$$

$$\Rightarrow \quad 18x = 36$$

$$\Rightarrow \quad x = 2$$

Using $9x^2 + 2y = 48$ we obtain:

$$36 + 2y = 48$$

$$\Rightarrow \quad 2y = 12$$

$$\Rightarrow \quad y = 6$$

The critical point is: $(2, 6)$

EXERCISES

Using the method of Lagrange Multipliers, find all critical points of the function $f(x, y)$ subject to the given constraint:

18–1) $f(x, y) = xy, \quad 2x + y = 4.$

18–2) $f(x, y) = 4xy + 50, \quad 3x - 4y = 24.$

18–3) $f(x, y) = x^2 + y^2, \quad x + 3y = 10.$

18–4) $f(x, y) = x^2 + 2y^2, \quad x + 4y = 36.$

18–5) $f(x, y) = x^2 + y^2, \quad 12x + 8y = 13.$

18–6) $f(x, y) = x^2 - y^2, \quad 3x + y = 8.$

18–7) $f(x, y) = 4x^2 - y^2, \quad 4x + y = 6.$

18–8) $f(x, y) = xy, \quad 9x^2 + 4y^2 = 72.$

18–9) $f(x, y) = xy, \quad 16x^2 + 25y^2 = 3200.$

18–10) $f(x, y) = xy, \quad x^2 + 36y^2 = 18.$

Using the method of Lagrange Multipliers, find all critical points of the function $f(x, y)$ subject to the given constraint:

18–11) $f(x, y) = x^2 - y^2, \quad x^2 + y^2 = 9.$

18–12) $f(x, y) = x^2 - 3y^2, \quad x^2 + y^2 = 4.$

18–13) $f(x, y) = x^2 - 3xy + y^2, \quad x^2 + y^2 = 8.$

18–14) $f(x, y) = x^2 + 18xy + 9y^2, \quad x^2 + 9y^2 = 2.$

18–15) $f(x, y) = xy^2, \quad 3x + 2y = 18.$

18–16) $f(x, y) = xy^4, \quad x + 30y = 75.$

18–17) $f(x, y) = 3x + 4y, \quad x^2 + y^2 = 100.$

18–18) $f(x, y) = x + 2y, \quad x^2 + y^2 = 45.$

18–19) $f(x, y) = 2x + y, \quad 5x^2 + 2y^2 = 130.$

18–20) $f(x, y) = 4x + 4y, \quad 16x^2 + 2y^2 = 9.$

Using the method of Lagrange Multipliers, find all critical points of the function $f(x,y)$ subject to the given constraint:

18–21) $f(x,y) = 9x^2 + 36xy + 16y^2, \quad 3x + 4y = 24.$

18–22) $f(x,y) = 25x^2 + 4xy + y^2, \quad 5x + y = 10.$

18–23) $f(x,y) = x^2 + 5xy + y^2, \quad x + y = 12.$

18–24) $f(x,y) = 4x^2 + 2xy + y^2, \quad 2x + y = 8.$

18–25) $f(x,y) = 4x^2 - 20xy + 25y^2, \quad 2x + 5y = 20.$

18–26) $f(x,y) = x^2 + 8xy + 4y^2, \quad x + 2y = 1.$

18–27) $f(x,y) = 6x + y, \quad 3x^2 + y = 6.$

18–28) $f(x,y) = x + 2y, \quad x^2 + 32y = 96.$

18–29) $f(x,y) = x + 3y, \quad x^2 + 18y = 63.$

18–30) $f(x,y) = 3x + y, \quad 75x^2 + 10y = 8.$

ANSWERS

18–1) $(1,2)$

18–2) $(4,-3)$

18–3) $(1,3)$

18–4) $(4,8)$

18–5) $\left(\frac{3}{4}, \frac{1}{2}\right)$

18–6) $(3,-1)$

18–7) $(2,-2)$

18–8) $(2,3), \quad (-2,-3), \quad (-2,3), \quad (2,-3).$

18–9) $(10,8), \quad (-10,-8), \quad (-10,8), \quad (10,-8).$

18–10) $\left(3, \frac{1}{2}\right), \quad \left(-3, -\frac{1}{2}\right), \quad \left(-3, \frac{1}{2}\right), \quad \left(3, -\frac{1}{2}\right).$

18–11) $(0, \pm 3)$, $(\pm 3, 0)$.

18–12) $(0, \pm 2)$, $(\pm 2, 0)$.

18–13) $(2, 2)$, $(2, -2)$, $(-2, 2)$, $(-2, -2)$.

18–14) $\left(1, \frac{1}{3}\right)$, $\left(1, -\frac{1}{3}\right)$, $\left(-1, \frac{1}{3}\right)$, $\left(-1, -\frac{1}{3}\right)$.

18–15) $(2, 6)$, $(6, 0)$.

18–16) $(15, 2)$, $(75, 0)$.

18–17) $(6, 8)$, $(-6, -8)$.

18–18) $(3, 6)$, $(-3, -6)$.

18–19) $(4, 5)$, $(-4, -5)$.

18–20) $\left(\frac{1}{4}, 2\right)$, $\left(-\frac{1}{4}, -2\right)$.

18–21) $(4, 3)$

18–22) $(1, 5)$

18–23) $(6, 6)$

18–24) $(2, 4)$

18–25) $(5, 2)$

18–26) $\left(\frac{1}{2}, \frac{1}{4}\right)$

18–27) $(1, 3)$

18–28) $(8, 1)$

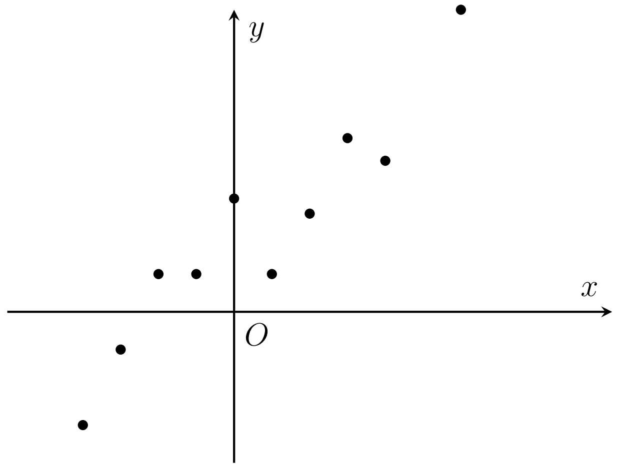
18–29) $(3, 3)$

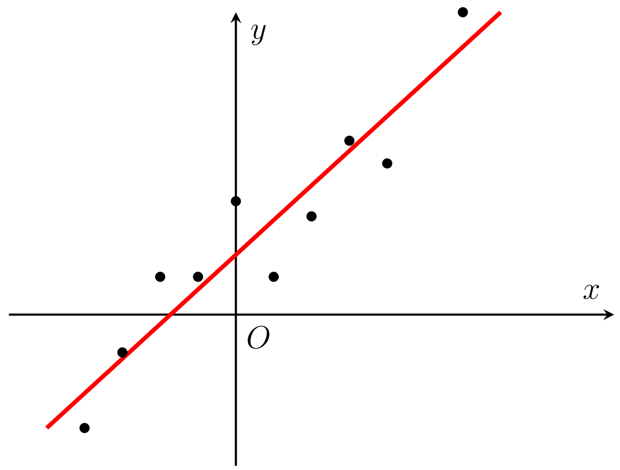
18–30) $\left(\frac{1}{5}, \frac{1}{2}\right)$

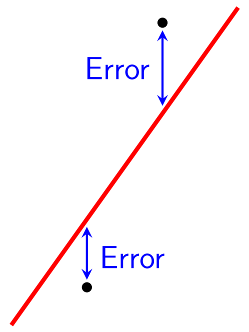
Chapter 19

Least Squares Line

We know that a line passes through two given points. If we have more than two points, they will not lie on a line unless specially chosen. Still, we can find a line that passes close to them.







Note that error is the vertical distance from the point to the line.

The least squares line is the line with minimum square error. We do not directly add errors, because if we did that, positive and negative errors would cancel each other.

If we are given the coordinates of n points:

x_1	x_2	$\cdots$	x_n
y_1	y_2	$\cdots$	y_n

and if we assume the equation of the line is:

$$y = ax + b$$

then, the total squared error is:

$$E = (ax_1 + b - y_1)^2 + (ax_2 + b - y_2)^2 + \cdots + (ax_n + b - y_n)^2$$

Note that, here, x_i and y_i are constant, a and b are variable. To minimize this function, we have to find partial derivatives and set them equal to zero.

$$\frac{\partial E}{\partial a} = 2(ax_1 + b - y_1)x_1 + \cdots + 2(ax_n + b - y_n)x_n = 0$$

$$\frac{\partial E}{\partial b} = 2(ax_1 + b - y_1) + \cdots + 2(ax_n + b - y_n) = 0$$

After simplifications, we obtain the system of equations:

$$\begin{aligned} (x_1^2 + \cdots + x_n^2)a + (x_1 + \cdots + x_n)b &= x_1y_1 + \cdots + x_ny_n \\ (x_1 + \cdots + x_n)a + nb &= y_1 + \cdots + y_n \end{aligned}$$

It is easier to express this system using sigma notation:

$$\begin{aligned} \left(\sum_{i=1}^{n} x_i^2\right) a + \left(\sum_{i=1}^{n} x_i\right) b &= \sum_{i=1}^{n} x_iy_i \\ \left(\sum_{i=1}^{n} x_i\right) a + nb &= \sum_{i=1}^{n} y_i \end{aligned}$$

The solution will give us the parameters of the least squares line:

$$y = ax + b$$

Example 19–1: Find the least squares line for the following points:

x	1	3	5	7
y	-1	2	2	3

Solution: Total squared error is:

$$E = (a+b+1)^2+(3a+b-2)^2+(5a+b-2)^2+(7a+b-3)^2$$

$$\frac{\partial E}{\partial a} = 2(a+b+1)+2(3a+b-2)3+2(5a+b-2)5+2(7a+b-3)7 = 0$$

$$\frac{\partial E}{\partial b} = 2(a+b+1)+2(3a+b-2)+2(5a+b-2)+2(7a+b-3) = 0$$

After simplifications, we obtain the system of equations:

$$\begin{aligned} 84a + 16b &= 36 \\ 16a + 4b &= 6 \end{aligned}$$

Multiply the second equation by 4 and then subtract from the first equation:

$$\begin{aligned} 84a + 16b &= 36 \\ 64a + 16b &= 24 \end{aligned}$$

$$20a = 12 \quad \Rightarrow \quad a = \frac{3}{5}$$

$$\Rightarrow \quad b = \frac{6 - 16a}{4} = -\frac{9}{10}$$

The least squares line is

$$y = \frac{3}{5}x - \frac{9}{10}$$

An alternative method is to use the formula directly:

x	1	3	5	7
y	-1	2	2	3

$$\begin{aligned} \sum_{i=1}^{n} x_i^2 &= 1^2+3^2+5^2+7^2 &= 84 \\ \sum_{i=1}^{n} x_i &= 1+3+5+7 &= 16 \\ \sum_{i=1}^{n} x_i y_i &= -1+6+10+21 &= 36 \\ \sum_{i=1}^{n} y_i &= -1+2+2+3 &= 6 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} 84a + 16b &= 36 \\ 16a + 4b &= 6 \end{aligned}$$

The rest of the solution is the same.

$$a = \frac{3}{5}, \qquad b = -\frac{9}{10}$$

EXERCISES

Find the equation of the least squares line using the following data:

19–1)

x	1	2	3
y	2	4	3

19–2)

x	0	2	4
y	-1	0	5

19–3)

x	1	2	3	4
y	-4	0	2	2

19–4)

x	1	2	3	4
y	5	3	1	0

19–5)

x	1	2	3	5
y	-1	0	2	4

19–6)

x	-1	0	2	3
y	6	3	0	-1

Find the equation of the least squares line using the following data:

19–7)

x	-2	-1	0	1	2
y	1	3	2	4	5

19–8)

x	-2	-1	0	1	2
y	8	5	0	1	-2

19–9)

x	0	1	2	3	4
y	10	6	2	0	0

19–10)

x	-1	1	2	3	5
y	6	0	1	-2	-4

19–11)

x	-2	0	2	4	6
y	1	3	2	4	5

19–12)

x	-3	-2	0	2	3
y	-5	-2	0	-1	3

ANSWERS

19–1) $y = \dfrac{1}{2}x + 2$

19–2) $y = \dfrac{3}{2}x - \dfrac{5}{3}$

19–3) $y = 2x - 5$

19–4) $y = -\dfrac{17}{10}x + \dfrac{13}{2}$

19–5) $y = \dfrac{9}{7}x - \dfrac{16}{7}$

19–6) $y = -\dfrac{17}{10}x + \dfrac{37}{10}$

19–7) $y = 0.9x + 3$

19–8) $y = -2.4x + 2.4$

19–9) $y = -2.6x + 8.8$

19–10) $y = -1.6x + 3.4$

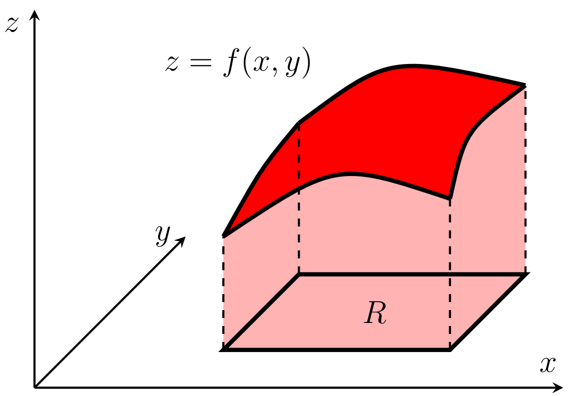
19–11) $y = 0.45x + 2.1$

19–12) $y = x - 1$

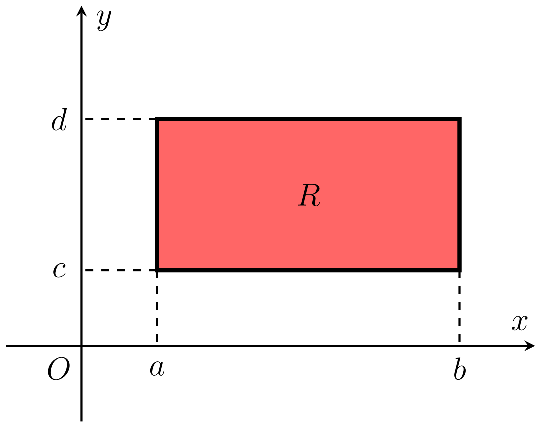
Chapter 20

Double Integrals

Consider the rectangle $R = [a, b] \times [c, d]$. We can think of the double integral of the function $f(x, y)$ over R as the volume under the graph of the function and above the $xy-$plane.



$$\text{Volume} = \iint_R f(x, y)\, dA$$



If $f(x, y)$ is continuous on R, we can evaluate the double integral as follows:

$$\begin{aligned} \iint_R f(x, y)\, dA &= \int_a^b \int_c^d f(x, y)\, dy\, dx \\ &= \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx \\ &= \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy \end{aligned}$$

Example 20–1: Evaluate the integral

$$\int_0^2 \int_0^3 \left(1 + xy^2\right) dy\, dx$$

Solution: Note that, when we are evaluating the integral with respect to y, we keep x constant.

$$\begin{aligned}
\int_0^2 \int_0^3 \left(1 + xy^2\right) dy\, dx &= \int_0^2 \left(\int_0^3 \left(1 + xy^2\right) dy \right) dx \\
&= \int_0^2 \left(y + \frac{xy^3}{3} \Big|_0^3 \right) dx \\
&= \int_0^2 (3 + 9x)\, dx \\
&= 3x + \frac{9x^2}{2} \Big|_0^2 \\
&= 6 + 18 \\
&= 24
\end{aligned}$$

We can change the order of integration. We have to change the order of the limits too.

$$\begin{aligned}
\int_0^2 \int_0^3 \left(1 + xy^2\right) dy\, dx &= \int_0^3 \int_0^2 \left(1 + xy^2\right) dx\, dy \\
&= \int_0^3 \left(\int_0^2 \left(1 + xy^2\right) dx \right) dy \\
&= \int_0^3 \left(x + \frac{x^2y^2}{2} \Big|_0^2 \right) dx \\
&= \int_0^3 (2 + 2y^2)\, dy \\
&= 2y + \frac{2y^3}{3} \Big|_0^3 \\
&= 6 + 18 \\
&= 24
\end{aligned}$$

Example 20–2: Evaluate the integral

$$\int_0^4 \int_2^6 \frac{5x\sqrt{x}}{4y}\,dy\,dx$$

Solution:

$$\begin{aligned}
\int_0^4 \int_2^6 \frac{5x\sqrt{x}}{4y}\,dy\,dx &= \frac{5}{4}\int_0^4 \int_2^6 x^{3/2}\cdot\frac{1}{y}\,dy\,dx \\
&= \frac{5}{4}\int_0^4 x^{3/2}\left(\ln y\Big|_2^6\right)dx \\
&= \frac{5}{4}\big(\ln 6 - \ln 2\big)\int_0^4 x^{3/2}\,dx \\
&= \frac{5}{4}\ln 3\cdot\frac{x^{5/2}}{5/2}\Bigg|_0^4 \\
&= \frac{5}{4}\cdot\frac{2}{5}\ln 3\cdot 4^{5/2} \\
&= \frac{1}{2}\ln 3\cdot 2^5 \\
&= 16\ln 3
\end{aligned}$$

Example 20–3: Evaluate the integral

$$\int_0^1 \int_2^3 2ye^{-3x}e^{y^2}\,dy\,dx$$

Solution:

$$\begin{aligned}
\int_0^1 \int_2^3 2ye^{-3x}e^{y^2}\,dy\,dx &= \int_0^1 e^{-3x}\left(\int_2^3 2ye^{y^2}\,dy\right)dx \\
&= \int_0^1 e^{-3x}\left(\int_4^9 e^u\,du\right)dx \\
&= \int_0^1 e^{-3x}\left(e^u\Big|_4^9\right)dx \\
&= \big(e^9 - e^4\big)\int_0^1 e^{-3x}\,dx \\
&= \big(e^9 - e^4\big)\left(\frac{e^{-3x}}{-3}\Bigg|_0^1\right) \\
&= \big(e^9 - e^4\big)\left(\frac{e^{-3}}{-3} - \frac{e^0}{-3}\right) \\
&= \frac{1}{3}\big(e^9 - e^4\big)\big(1 - e^{-3}\big)
\end{aligned}$$

EXERCISES

Evaluate the following double integrals:

20–1) $\displaystyle\int_0^2 \int_0^1 \big(8y^3 + xy\big)\, dy\, dx$

20–2) $\displaystyle\int_0^4 \int_0^2 \big(2 + 3x^2 + 4y\big)\, dy\, dx$

20–3) $\displaystyle\int_1^4 \int_0^1 \sqrt{xy}\, dy\, dx$

20–4) $\displaystyle\int_0^1 \int_9^{15} x^2\sqrt{x}\, dy\, dx$

20–5) $\displaystyle\int_3^5 \int_6^9 \frac{x}{y^2}\, dy\, dx$

20–6) $\displaystyle\int_4^6 \int_5^{10} \frac{x}{y}\, dy\, dx$

20–7) $\displaystyle\int_1^2 \int_0^1 (1 + x^2)e^y\, dy\, dx$

20–8) $\displaystyle\int_0^1 \int_0^1 e^{y-2x}\, dy\, dx$

20–9) $\displaystyle\int_1^2 \int_2^7 \frac{e^{5x}}{y}\, dy\, dx$

20–10) $\displaystyle\int_1^3 \int_2^4 \frac{1}{(xy)^3}\, dy\, dx$

Evaluate the following double integrals:

20–11) $\displaystyle\int_0^1 \int_0^4 \big(x^3 + 1\big)x^2 y\, dy\, dx$

20–12) $\displaystyle\int_1^9 \int_0^3 \sqrt{\frac{y+1}{x}}\, dy\, dx$

20–13) $\displaystyle\int_1^5 \int_0^2 \frac{5y + 3}{\sqrt{2x - 1}}\, dy\, dx$

20–14) $\displaystyle\int_7^{11} \int_1^3 \frac{y}{y^2 + 6}\, dy\, dx$

20–15) $\displaystyle\int_0^1 \int_1^5 \frac{ye^x}{e^x + 1}\, dy\, dx$

20–16) $\displaystyle\int_0^1 \int_0^1 xe^{xy}\, dy\, dx$

20–17) $\displaystyle\int_3^9 \int_1^2 \frac{\ln x}{xy}\, dy\, dx$

20–18) $\displaystyle\int_0^2 \int_2^3 \frac{24x}{y^2 e^{x^2}}\, dy\, dx$

20–19) $\displaystyle\int_0^2 \int_0^2 \frac{18x^3y^3}{\big(x^4 + 2\big)^2}\, dy\, dx$

20–20) $\displaystyle\int_0^1 \int_0^1 x\big(1 + xy\big)^4\, dy\, dx$

ANSWERS

20–1) 5

20–2) 176

20–3) $\frac{28}{9}$

20–4) $\frac{12}{7}$

20–5) $\frac{4}{9}$

20–6) $10\ln 2$

20–7) $\frac{10}{3}\left(e-1\right)$

20–8) $\frac{1}{2}\left(e-1\right)\left(1-e^{-2}\right)$

20–9) $\frac{1}{5}\left(e^{10}-e^{5}\right)\ln\frac{7}{2}$

20–10) $\frac{1}{24}$

20–11) 4

20–12) $\frac{56}{3}$

20–13) 32

20–14) $2\ln\frac{15}{7}$

20–15) $12\ln\frac{e+1}{2}$

20–16) $e-2$

20–17) $\frac{3}{2}\ln^2 3\cdot\ln 2$

20–18) $2\left(1-e^{-4}\right)$

20–19) 8

20–20) $\frac{19}{10}$

Chapter 21

Separable Differential Equations

Example 21–1: Solve the differential equation

$$y' = -4y^2x^3$$

Solution: First we have to write this equation in the form:

$$\frac{dy}{dx} = -4y^2x^3$$

$$-\frac{dy}{y^2} = 4x^3dx$$

Now we can solve this by integrating both sides:

$$\int -y^{-2}dy = \int 4x^3dx$$

$$y^{-1} = x^4 + c$$

$$\frac{1}{y} = x^4 + c$$

$$y = \frac{1}{x^4 + c}$$

Example 21–2: Solve the intial value problem

$$y' = x^2e^{-y}, \quad y(2) = 0$$

Solution: This time, we will solve the equation and then also find the integration constant:

$$\frac{dy}{dx} = x^2e^{-y}$$

$$\int e^y dy = \int x^2dx$$

$$e^y = \frac{x^3}{3} + c$$

$$x = 2 \quad \Rightarrow \quad y = 0$$

$$e^0 = \frac{2^3}{3} + c \quad \Rightarrow \quad c = -\frac{5}{3}$$

$$y = \ln\left(\frac{x^3 - 5}{3}\right)$$

EXERCISES

Find the solution of the following differential equations:

21–1) $y' = 1 + \dfrac{1}{x}$

21–2) $y' = -\dfrac{x}{2y}$

21–3) $y' = x^2y^2$

21–4) $y^2y' = 4x$

21–5) $xy' = -y^2$

21–6) $y' = 2xe^{-y}$

21–7) $yy' = 1 - 2e^{-4x}$

21–8) $yy' = 6e^{3x-y}$

21–9) $y' = \dfrac{1}{5-2y}$

21–10) $y' = \dfrac{y}{x}$

21–11) $y' = x^2y$

21–12) $y' = -y$

Find the solution of the following initial value problems:

21–13) $y' = 2, \quad y(0) = 9.$

21–14) $y' = x\sqrt{x}, \quad y(1) = 1.$

21–15) $y' = -2(x-2)^{-3} + 12, \quad y(3) = 37.$

21–16) $y' = 24x^2, \quad y(0) = 4.$

21–17) $y' = e^{-x}, \quad y(0) = 0.$

21–18) $y' = \dfrac{1}{2}(1+x)^{-1/2} + 1, \quad y(3) = 5.$

21–19) $y' = \dfrac{5}{(1-x)^{3/2}}, \quad y(0) = 15.$

21–20) $y' = \dfrac{1}{x}, \quad y(1) = 4.$

21–21) $y' = -6e^{-2x} - e^{-x}, \quad y(0) = 5.$

21–22) $y' = \dfrac{2x}{1+x^2}, \quad y(0) = 0.$

21–23) $y' = -12\,\dfrac{\sqrt{1+y}}{x^3}, \quad y(1) = 8.$

21–24) $yy' = 2e^{-y^2}, \quad y(0) = 0.$

ANSWERS

21–1) $y = x + \ln x + c$

21–2) $y^2 + \dfrac{x^2}{2} = c$

21–3) $y = -\dfrac{3}{x^3 + 3c}$

21–4) $y = (6x^2 + c)^{1/3}$

21–5) $y = \dfrac{1}{\ln x + c}$

21–6) $y = \ln(x^2 + c)$

21–7) $y = \sqrt{2x + e^{-4x} + c}$

21–8) $y = \ln(2e^{3x} + c)$

21–9) $5y - y^2 = x + c$

21–10) $y = kx$

21–11) $y = ke^{x^3/3}$

21–12) $y = ke^{-x}$

21–13) $y = 2x + 9$

21–14) $y = \dfrac{2}{5}x^{5/2} + \dfrac{3}{5}$

21–15) $y = \dfrac{1}{(x-2)^2} + 12x$

21–16) $y = 8x^3 + 4$

21–17) $y = 1 - e^{-x}$

21–18) $y = \sqrt{1+x} + x$

21–19) $y = \dfrac{10}{\sqrt{1-x}} + 5$

21–20) $y = \ln x + 4$

21–21) $y = 3e^{-2x} + e^{-x} + 1$

21–22) $y = \ln|1 + x^2|$

21–23) $y = \dfrac{9}{x^4} - 1$

21–24) $y = \sqrt{\ln(4x+1)}$

Chapter 22

First Order Linear Differential Equations

A differential equation that can be written in the form

$$y' + p(x)y = r(x)$$

is called a first order linear differential equation. Here $y = y(x)$ and $p(x)$ and $q(x)$ are functions of x. For example

$$y' + x^2 y = e^{-x}$$

$$y' - \frac{y}{x} = x^4 + x$$

$$y' + 5y = \ln x$$

are first order linear differential equations. But the following are NOT:

$$y' + x^3 y = y^2$$

$$yy' - xy = x^2 + 1$$

$$y' + 5e^y = \frac{1}{x}$$

These equations are not in the form given in the definition.

Method of Solution:
To solve a first order linear differential equation:

- Find $q(x) = \displaystyle\int p(x)\,dx$.
- Multiply both sides of the equation by $e^{q(x)} = e^{\int p(x)\,dx}$. Obtain:

$$e^{q(x)}y' + e^{q(x)}p(x)y = e^{q(x)}r(x)$$

- The left hand side can be written as the derivative of a product:

$$\left[e^{q(x)}y\right]' = e^{q(x)}r(x)$$

- Integrate both sides:

$$e^{q(x)}y = \int e^{q(x)}r(x)dx$$

  Do not forget the integration constant.

Example 22–1: Solve $y' + \dfrac{2}{x}\,y = 4x$.

Solution: Here, $p(x) = \dfrac{2}{x}$ and $r(x) = 4x$.

$$\int \frac{2}{x}\,dx = 2\ln x$$

$$e^{2\ln x} = e^{\ln x^2} = x^2$$

Multiply both sides by x^2:

$$x^2y' + 2xy = 4x^3$$

$$\left[x^2y\right]' = 4x^3$$

$$x^2y = \int 4x^3dx$$

$$x^2y = x^4 + c$$

$$y = x^2 + \frac{c}{x^2}$$

Example 22–2: Solve the initial value problem:

$$y' - y = e^{2x}, \quad y(0) = 5$$

Solution: Here, $p(x) = -1$ and $r(x) = e^{2x}$.

$$\int -dx = -x$$

Multiply both sides by e^{-x}:

$$e^{-x}y' - xe^{-x}y = e^{x}$$

$$\left[e^{-x}y\right]' = e^{x}$$

$$e^{-x}y = \int e^{x}\,dx$$

$$e^{-x}y = e^{x} + c$$

$$y = e^{2x} + ce^{x}$$

In this case, we have to find the integration constant using the initial condition.

$$x = 0 \quad \Rightarrow \quad y = 5$$

$$5 = 1 + c \quad \Rightarrow \quad c = 4$$

$$y = e^{2x} + 4e^{x}$$

EXERCISES

Find the solution of the following differential equations:

22–1) $y' - y = xe^x$

22–2) $y' + \dfrac{2}{x}\, y = \dfrac{e^{-x}}{x^2}$

22–3) $y' - 2xy = 3x^2 e^{x^2}$

22–4) $y' - \dfrac{y}{x} = 4x^4$

22–5) $y' + \dfrac{y}{2x} = 6$

22–6) $y' + 2y = \dfrac{5}{xe^{2x}}$

22–7) $y' + \left(1 + \dfrac{1}{x}\right) y = \left(\dfrac{2}{x} + 6\right) e^{-x}$

22–8) $y' + y = \dfrac{1}{x^2 e^x}$

22–9) $y' + 3x^2 y = 7e^{-x^3}$

22–10) $y' - \dfrac{y}{2x} = 3x^{5/2}$

22–11) $y' - 2y = 15\sqrt{x}\, e^{2x}$

22–12) $x^3 y' + x^2 y = -1$

Find the solution of the following initial value problems:

22–13) $xy' + 4y = \dfrac{2}{x^4}, \quad y(1) = 0.$

22–14) $y' + 3y = 8e^x, \quad y(0) = 6.$

22–15) $xy' - 2y = 2, \quad y(2) = 7.$

22–16) $y' + 4y = 12e^{-4x}, \quad y(0) = 2.$

22–17) $y' - \dfrac{y}{2x} = \dfrac{1}{x}, \quad y(1) = 4.$

22–18) $x^2 y' - 3xy = x^4, \quad y(5) = 0.$

22–19) $xy' + 2y = \dfrac{2}{1 + x^2}, \quad y(1) = \ln 6.$

22–20) $y' - \dfrac{y}{x} = 15x(3x+1)^4, \quad y\left(\dfrac{1}{3}\right) = 12.$

22–21) $y' - \dfrac{2y}{x} = 4x \ln x, \quad y(1) = 8.$

22–22) $xy' + 3y = 10x^2, \quad y(2) = 15.$

22–23) $y' + y = \dfrac{1}{1 + e^x}, \quad y(0) = 0.$

22–24) $y' + \dfrac{y}{x \ln x} = \dfrac{8}{x \ln x}, \quad y(e) = 11.$

ANSWERS

22–1) $y = \left(\dfrac{x^2}{2} + c\right) e^x$

22–2) $y = \dfrac{c - e^{-x}}{x^2}$

22–3) $y = \left(x^3 + c\right)e^{x^2}$

22–4) $y = x^5 + cx$

22–5) $y = 4x + \dfrac{c}{\sqrt{x}}$

22–6) $y = \left(5\ln x + c\right)e^{-2x}$

22–7) $y = \left(\dfrac{c}{x} + 2 + 3x\right) e^{-x}$

22–8) $y = \left(c - \dfrac{1}{x}\right) e^{-x}$

22–9) $y = \left(7x + c\right)e^{-x^3}$

22–10) $y = \left(x^3 + c\right)\sqrt{x}$

22–11) $y = \left(10x\sqrt{x} + c\right)e^{2x}$

22–12) $y = \dfrac{1}{x^2} + \dfrac{c}{x}$

22–13) $y = \dfrac{2\ln x}{x^4}$

22–14) $y = 2e^x + 4e^{-3x}$

22–15) $y = 2x^2 - 1$

22–16) $y = \left(12x + 2\right)e^{-4x}$

22–17) $y = 6\sqrt{x} - 2$

22–18) $y = x^3 \ln\left(\dfrac{x}{5}\right)$

22–19) $y = \dfrac{\ln\left(3 + 3x^2\right)}{x^2}$

22–20) $y = x\left(3x + 1\right)^5 + 4x$

22–21) $y = 2x^2\left(\ln^2 x + 4\right)$

22–22) $y = 2x^2 + \dfrac{56}{x^3}$

22–23) $y = e^{-x}\ln\left(\dfrac{1 + e^x}{2}\right)$

22–24) $y = 8 + \dfrac{3}{\ln x}$

Chapter 23

Second Order Differential Equations

A differential equation that can be written in the form

$$y'' + ay' + by = 0$$

where a and b are constants, is called a second order linear homogeneous differential equation with constant coefficients. Some examples are:

$$y'' + 5y' + 4y = 0$$

$$y'' - \frac{3}{2}y' + \frac{1}{2}y = 0$$

$$y'' + 9y = 0$$

$$4y'' - 9y' + 5y = 0$$

The last equation is not exactly in the format given above, but we can easily rewrite it as:

$$y'' - \frac{9}{4}y' + \frac{5}{4}y = 0$$

Method of Solution:

- Start with the assumption $y = e^{rx}$ where r is a real number.
- Insert $y = e^{rx}$ in the equation and obtain:

$$r^2 + ar + b = 0$$

- Solve this quadratic equation.
- If there are two distinct roots r_1 and r_2, the solution is:

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

- If r is a double root

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

Note that we do not consider complex roots in this course.

c_1 and c_2 are arbitrary constants, in other words, you can choose them as you wish and still the function y will satisfy the given differential equation.

Example 23–1: Solve $y'' - 5y' + 6y = 0$.

Solution: Let's start with the assumption $y = e^{rx}$. In that case:

$$y' = re^{rx}$$

$$y'' = r^2e^{rx}$$

Insert these in the equation to obtain:

$$r^2e^{rx} - 5re^{rx} + 6e^{rx} = 0$$

$$e^{rx}\Big(r^2 - 5r + 6\Big) = 0$$

We know that $e^{rx} \neq 0$ so

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0$$

We find two roots: $r = 2$ or $r = 3$. There are two solutions of the differential equation, e^{2x} and e^{3x}.

The equation is linear and homogeneous, so if we multiply a solution by a constant, or add two solutions, the result is again a solution.

Therefore the general solution is:

$$y = c_1e^{2x} + c_2e^{3x}$$

Example 23–2: Solve $y'' - 9y = 0$.

Solution: Once again,

$$y = e^{rx}$$

$$y' = re^{rx}$$

$$y'' = r^2e^{rx}$$

Insert these in the equation to obtain:

$$r^2e^{rx} - 9e^{rx} = 0$$

$$r^2 - 9 = 0$$

$$(r-3)(r-3) = 0$$

We find two roots: $r = -3$ or $r = 3$. The general solution is:

$$y = c_1e^{3x} + c_2e^{-3x}$$

Example 23–3: Solve $y'' + 15y' = 0$.

Solution: Once again,

$$y = e^{rx}$$
$$y' = re^{rx}$$
$$y'' = r^2e^{rx}$$

Insert these in the equation to obtain:

$$r^2e^{rx} + 15re^{rx} = 0$$
$$r^2 + 15r = 0$$
$$r(r + 15) = 0$$

We find two roots: $r = 0$ or $r = -15$. We know that $e^0 = 1$, this means one solution is a constant. The general solution is:

$$y = c_1 + c_2e^{-15x}$$

Example 23–4: Solve $y'' - 4y' + 4y = 0$.

Solution:

$$y = e^{rx}$$
$$y' = re^{rx}$$
$$y'' = r^2e^{rx}$$

Insert these in the equation to obtain:

$$r^2e^{rx} - 4re^{rx} + 4e^{rx} = 0$$
$$r^2 - 4r + 4 = 0$$
$$(r - 2)^2 = 0$$

We find $r = 2$. This is a double root. The general solution is:

$$y = c_1e^{2x} + c_2xe^{2x}$$

Example 23–5: Solve the initial value problem

$$4y'' - 8y' + 3y = 0, \quad y(0) = 4, \quad y'(0) = 1$$

Solution: First, we will solve the equation and then we will determine the constants using the given initial conditions.

$$4r^2e^{rx} - 8re^{rx} + 3e^{rx} = 0$$

$$4r^2 - 8r + 3 = 0$$

$$(2r-1)(2r-3) = 0$$

We find two roots: $r = \dfrac{1}{2}$ and $r = \dfrac{3}{2}$. The general solution is:

$$y = c_1 e^{\frac{1}{2}x} + c_2 e^{\frac{3}{2}x}$$

The derivative of y is:

$$y' = \frac{1}{2}c_1 e^{\frac{1}{2}x} + \frac{3}{2}c_2 e^{\frac{3}{2}x}$$

$y(0) = 4$ means that when $x = 0$, $y = 4$. In other words $c_1 + c_2 = 4$. Similarly:

$$y'(0) = 1 \quad \Rightarrow \quad \frac{1}{2}c_1 + \frac{3}{2}c_2 = 1 \quad \Rightarrow \quad c_1 + 3c_2 = 2.$$

The solution of the system

$$\begin{aligned} c_1 + c_2 &= 4 \\ c_1 + 3c_2 &= 2 \end{aligned}$$

is $c_1 = 5,\ c_2 = -1$ therefore

$$y = 5e^{\frac{1}{2}x} - e^{\frac{3}{2}x}$$

Example 23–6: Solve the initial value problem

$$y'' - 12y' + 36y = 0, \quad y(0) = 3, \quad y'(0) = 13$$

Solution:

$$r^2e^{rx} - 12re^{rx} + 36e^{rx} = 0$$

$$r^2 - 12r + 36 = 0$$

$$(r-6)^2 = 0$$

We find $r = 6$. This is a double root. The general solution is:

$$y = c_1 e^{6x} + c_2 x e^{6x}$$

The derivative of y is:

$$y' = 6c_1 e^{6x} + c_2 e^{6x} + 6c_2 x e^{6x}$$

$$y(0) = 3 \quad \Rightarrow \quad c_1 = 3.$$

$$y'(0) = 13 \quad \Rightarrow \quad 6c_1 + c_2 = 13.$$

The solution of this set of equations is $c_1 = 3,\ c_2 = -5$ therefore

$$y = 3e^{6x} - 5xe^{6x}$$

EXERCISES

Find the general solution of the following differential equations:

23–1) $y'' - 8y' + 15y = 0$

23–2) $y'' - 9y' + 8y = 0$

23–3) $y'' - 2y' - 8y = 0$

23–4) $y'' - 9y = 0$

23–5) $y'' - 64y = 0$

23–6) $y'' - 6y' = 0$

23–7) $y'' + 11y' = 0$

23–8) $y'' - y' + \frac{3}{16}y = 0$

23–9) $25y'' - 50y' + 16y = 0$

23–10) $y'' - 8y' + 16y = 0$

23–11) $y'' + 18y' + 81y = 0$

23–12) $9y'' - 12y' + 4y = 0$

Find the solution of the following initial value problems:

23–13) $y'' - 7y' + 10y = 0, \quad y(0) = 7, \quad y'(0) = 26.$

23–14) $y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = -4.$

23–15) $y'' - y' = 0, \quad y(0) = 2, \quad y'(0) = 1.$

23–16) $49y'' - 4y = 0, \quad y(0) = -7, \quad y'(0) = 2.$

23–17) $16y'' - 48y' + 35y = 0, \quad y(0) = 12, \quad y'(0) = 19.$

23–18) $4y'' - 20y' + 25y = 0, \quad y(0) = 0, \quad y'(0) = 11.$

23–19) $y'' + 3y' - 4y = 0, \quad y(0) = -6, \quad y'(0) = 29.$

23–20) $y'' - 16y = 0, \quad y(0) = 15, \quad y'(0) = 12.$

23–21) $y'' - 20y' + 100y = 0, \quad y(0) = 0, \quad y'(0) = 4.$

23–22) $y'' - 14y' + 24y = 0, \quad y(0) = 3, \quad y'(0) = 36.$

23–23) $25y'' - 10y' + y = 0, \quad y(0) = 5, \quad y'(0) = -14.$

23–24) $y'' - 8y' = 0, \quad y(0) = 1, \quad y'(0) = 8.$

ANSWERS

23–1) $y = c_1e^{3x} + c_2e^{5x}$

23–2) $y = c_1e^{x} + c_2e^{8x}$

23–3) $y = c_1e^{-2x} + c_2e^{4x}$

23–4) $y = c_1e^{3x} + c_2e^{-3x}$

23–5) $y = c_1e^{8x} + c_2e^{-8x}$

23–6) $y = c_1 + c_2e^{6x}$

23–7) $y = c_1 + c_2e^{-11x}$

23–8) $y = c_1e^{\frac{1}{4}x} + c_2e^{\frac{3}{4}x}$

23–9) $y = c_1e^{\frac{2}{5}x} + c_2e^{\frac{8}{5}x}$

23–10) $y = c_1e^{4x} + c_2xe^{4x}$

23–11) $y = c_1e^{-9x} + c_2xe^{-9x}$

23–12) $y = c_1e^{\frac{2}{3}x} + c_2xe^{\frac{2}{3}x}$

23–13) $y = 3e^{2x} + 4e^{5x}$

23–14) $y = e^{-x} - 3xe^{-x}$

23–15) $y = 1 + e^{x}$

23–16) $y = -7e^{-\frac{2}{7}x}$

23–17) $y = 8e^{\frac{7}{4}x} + 4e^{\frac{5}{4}x}$

23–18) $y = 11xe^{\frac{5}{2}x}$

23–19) $y = e^{x} - 7e^{-4x}$

23–20) $y = 9e^{4x} + 6e^{-4x}$

23–21) $y = 4xe^{10x}$

23–22) $y = 3e^{12x}$

23–23) $y = 5e^{\frac{1}{5}x} - 15xe^{\frac{1}{5}x}$

23–24) $y = e^{8x}$

Chapter 24

Sequences and Difference Equations

A sequence is an ordered list of numbers with infinitely many terms. For example

$$\{1, 3, 5, 7, 9, \dots\}$$

is an arithmetic sequence. We can also express this using the general term:

$$\{a_n = 2n - 1\}_{n=1}^{\infty} \quad \text{or} \quad a_n = 2n - 1, \quad n = 1, 2, 3, \dots$$

Here, a_n denote terms of the series. Another example is:

$$\{1, 2, 4, 8, 16, \dots\}$$

This is a geometric sequence.

$$\{a_n = 2^n\}_{n=0}^{\infty} \quad \text{or} \quad a_n = 2^n, \quad n = 0, 1, 2, \dots$$

There are many different types of sequences

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$$

$$a_n = \frac{1}{n}, \quad n = 1, 2, 3, \dots$$

Example 24–1: Find the general term of the following sequences:

a) $\{1, -1, 1, -1, 1, \dots\}$

b) $\{5, 10, 15, 20, 25, \dots\}$

c) $\{8, 13, 18, 23, 28, \dots\}$

d) $\{1, -2, 4, -8, 16, \dots\}$

e) $\left\{\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots\right\}$

Solution:

a) $a_n = (-1)^n, \qquad n = 0, 1, 2, \dots$

b) $a_n = 5n, \qquad n = 1, 2, 3, \dots$

c) $a_n = 5n + 3, \qquad n = 1, 2, 3, \dots$

d) $a_n = (-2)^n, \qquad n = 0, 1, 2, \dots$

e) $a_n = \dfrac{1}{n^2}, \qquad n = 2, 3, 4, \dots$

Arithmetic Sequences: An arithmetic sequence is a sequence where the difference between the terms is fixed, for example:

$$1,\ 2,\ 3,\ 4,\ \ldots$$

$$20,\ 25,\ 30,\ 35,\ \ldots$$

$$74,\ 174,\ 274,\ 374,\ \ldots$$

are arithmetic sequences. The general form is:

$$a_1,\ a_1+d,\ a_1+2d,\ a_1+3d,\ \ldots$$

where a_1 is the first term and d denotes the difference. The n$^{\text{th}}$ term is:

$$a_n = a_1 + (n-1)d$$

We can find the sum of the first n positive integer using the following trick:

$$\begin{aligned} S &= 1+2+3+4+\cdots+n \\ S &= n+(n-1)+(n-2)+(n-3)+\cdots+1 \end{aligned}$$

Add them side by side to obtain:

$$2S = (n+1)+(n+1)+(n+1)+(n+1)+\cdots+(n+1)$$

$$2S = n(n+1) \quad \Rightarrow \quad S = \frac{n(n+1)}{2}$$

The same trick for a general arithmetic sequence gives:

$$2S = (a_1+a_n)+(a_1+a_n)+\cdots+(a_1+a_n)$$

$$2S = n((a_1+a_n)) \quad \Rightarrow \quad S = \frac{a_1+a_n}{2}\,n$$

In other words, total is equal to the average term times the number of terms.

Example 24–2: An arithmetic sequence is given by $a_n = 12+5n$. Find a_1, a_2 and a_{100}.

Solution: The difference is 5. Let's rewrite the equation as:

$$a_n = 17+5(n-1)$$

$$a_1 = 17$$

$$a_2 = 22$$

$$a_{100} = 512$$

Example 24–3: Consider the arithmetic sequence $4,\ 12,\ 20,\ldots$. If $a_n = 604$, what is n?

Solution: The difference is 8. Let's write the general term as:

$$a_n = 4+8(n-1)$$

$$604 = 4+8(n-1) \quad \Rightarrow \quad 604 = 8n-4$$

$$n = \frac{608}{8} = 76$$

Example 24–4: Find the sum of $7+10+13+16+\cdots+73$.

Solution: The difference is 3. Let's write the general term as:

$$a_n = 7+3(n-1)$$

$$73 = 7+3(n-1) \quad \Rightarrow \quad n = 23$$

Therefore the sum is:

$$S = \frac{7+73}{2}\cdot 23 = 1120$$

Difference Equations: A linear first order difference equation with constant coefficients is:

$$x_{n+1} = ax_n + b, \quad n = 0, 1, 2, \ldots$$

Here, a and b are constants. If $b = 0$, we obtain the homogeneous equation

$$x_{n+1} = ax_n, \quad n = 0, 1, 2, \ldots$$

The homogeneous equations are easy:

$$\begin{aligned} x_1 &= ax_0 \\ x_2 &= ax_1 = a^2x_0 \\ x_3 &= ax_2 = a^3x_0 \\ x_4 &= ax_3 = a^4x_0 \\ &\vdots \\ x_n &= ax_{n-1} = a^nx_0 \end{aligned}$$

In other words,

$$x_{n+1} = ax_n \quad \Rightarrow \quad x_n = a^nx_0$$

Let's try to solve the nonhomogeneous equation $x_{n+1} = ax_n + b$ in the same way:

$$\begin{aligned} x_1 &= ax_0 + b \\ x_2 &= ax_1 + b = a^2x_0 + ab + b \\ x_3 &= ax_2 + b = a^3x_0 + a^2b + ab + b \\ &\vdots \\ x_n &= a^nx_0 + a^{n-1}b + \cdots + a^2b + ab + b \end{aligned}$$

$$x_{n+1} = ax_n + b \quad \Rightarrow \quad x_n = a^nx_0 + b\big(1 + a + a^2 + a^3 + \cdots + a^{n-1}\big)$$

Consider the following identities:

$$\begin{aligned} \big(1-a\big)\big(1+a\big) &= \big(1-a^2\big) \\ \big(1-a\big)\big(1+a+a^2\big) &= \big(1-a^3\big) \\ \big(1-a\big)\big(1+a+a^2+a^3\big) &= \big(1-a^4\big) \\ &\vdots \end{aligned}$$

We can see that for the general case, the formula is:

$$\big(1-a\big)\big(1+a+a^2+a^3+\cdots+a^{n-1}\big) = \big(1-a^n\big)$$

We can easily show this by multiplying the expressions on the left and simplifying. If we rearrange this formula, we obtain:

$$1 + a + a^2 + a^3 + \cdots + a^{n-1} = \frac{1-a^n}{1-a} = \frac{a^n-1}{a-1}, \qquad a \neq 1$$

If $a = 1$ we obtain:

$$1 + a + a^2 + a^3 + \cdots + a^{n-1} = n$$

Now, we can express the solution of the nonhomogeneous difference equation

$$x_{n+1} = ax_n + b, \quad n = 0, 1, 2, \ldots$$

as follows:

- If $a \neq 1$:
$$x_n = a^nx_0 + b\,\frac{a^n-1}{a-1}$$
- If $a = 1$:
$$x_n = x_0 + nb$$

Example 24–5: Find the solution of the difference equation:

$$x_{n+1} = 2x_n + 5, \quad x_0 = 6.$$

Solution: If we find the first few terms, we see that:

$$\begin{aligned} x_1 &= 2x_0 + 5 \\ x_2 &= 2x_1 + 5 = 2^2 x_0 + 2 \cdot 5 + 5 \\ x_3 &= 2x_2 + 5 = 2^3 x_0 + 2^2 \cdot 5 + 2 \cdot 5 + 5 \\ &\vdots \\ x_n &= 2^n \cdot 6 + 2^{n-1} \cdot 5 + \cdots + 2^2 \cdot 5 + 2 \cdot 5 + 5 \end{aligned}$$

$$x_n = 2^n \cdot 6 + 5 \cdot \frac{2^n - 1}{2 - 1}$$

Alternatively, we can directly see that here $a = 2,\ b = 5$ and $x_0 = 6$ so using the formula we obtain:

$$\begin{aligned} x_n &= 2^n \cdot 6 + 5 \cdot \frac{2^n - 1}{2 - 1} \\ &= 2^n \cdot 6 + 5 \cdot \left(2^n - 1\right) \\ &= 11 \cdot 2^n - 5 \end{aligned}$$

Example 24–6: Find the solution of the difference equation:

$$x_{n+1} = 4x_n - 13, \quad x_0 = 7.$$

Solution: $a = 5,\ b = -13$ and $x_0 = 7$ so using the formula:

$$\begin{aligned} x_n &= 4^n \cdot 7 - 13 \cdot \frac{4^n - 1}{4 - 1} \\ &= 4^n \cdot 7 - \frac{13}{3} \cdot \left(4^n - 1\right) \\ &= \frac{8}{3} \cdot 4^n + \frac{13}{3} \\ &= \frac{2 \cdot 4^{n+1} + 13}{3} \end{aligned}$$

Example 24–7: Find the solution of the difference equation:

$$x_{n+1} = x_n + 8, \quad x_0 = 33.$$

Solution: Here $a = 1$, so we have to use the second formula:

$$x_n = 33 + 8n$$

Actually, we can solve this easily without any formulas:

$$\begin{aligned} x_1 &= 33 + 8 \\ x_2 &= x_1 + 8 = 33 + 2 \cdot 8 \\ x_3 &= x_2 + 8 = 33 + 3 \cdot 8 \\ &\vdots \\ x_n &= x_{n-1} + 8 = 33 + n \cdot 8 \end{aligned}$$

Second Order Difference Equations: A second order linear homogeneous difference equation with constant coefficients is:

$$x_{n+2} + ax_{n+1} + x_n = 0, \quad n = 0, 1, 2, \ldots$$

where a and b are constants. We can try a solution of the form $x_n = r^n$. Then, we obtain:

$$r^{n+2} + ar^{n+1} + br^n = 0$$

$$r^2 + ar + b = 0$$

Note that, for this type of equation:

- A multiple of a solution is also a solution.
- Sum of two solutions is also a solution.

If there are two distinct roots r_1 and r_2, the general solution is:

$$x_n = c_1 r_1^n + c_2 r_2^n$$

If there is a double root r, the general solution is:

$$x_n = c_1 r^n + c_2 n r^n$$

(We do not consider complex roots in this course.)

If two initial conditions are given, we can determine c_1 and c_2. This method is very similar to the one we used for second order linear homogeneous differential equations.

Example 24–8: Find the solution of the difference equation:

$$x_{n+2} - 7x_{n+1} + 12x_n = 0, \quad x_0 = 4, \quad x_1 = 7.$$

Solution: Let's insert $x_n = r^n$ in the equation:

$$r^{n+2} - 7r^{n+1} + 12r^n = 0$$

$$r^2 - 7r + 12r = 0$$

$$(r-3)(r-4) = 0$$

$r = 3$ or $r = 4$. We have two distinct roots.

$$x_n = c_1 3^n + c_2 4^n$$

$$x_0 = 4 \quad \Rightarrow \quad c_1 + c_2 = 4$$

$$x_1 = 7 \quad \Rightarrow \quad 3c_1 + 4c_2 = 7$$

We have to solve this system of two equations in two unknowns.

$$\begin{aligned} 3c_1 + 3c_2 &= 12 \\ 3c_1 + 4c_2 &= 7 \end{aligned}$$

Subtracting these, we obtain $-c_2 = 5, c_2 = -5$ and $c_1 = 4 - c_2 = 9$. Therefore the solution of the difference equation is:

$$x_n = 9 \cdot 3^n - 5 \cdot 4^n$$

Example 24–9: Find the solution of the difference equation:

$$x_{n+2} - 12x_{n+1} + 36x_n = 0, \quad x_0 = 5, \quad x_1 = 18.$$

Solution: Let's insert $x_n = r^n$ in the equation:

$$r^{n+2} - 12r^{n+1} + 36r^n = 0$$

$$r^2 - 12r + 36 = 0$$

$$(r-6)^2 = 0$$

$r = 6$ is a double root.

$$x_n = c_1 6^n + c_2 n 6^n$$

$$x_0 = 5 \quad \Rightarrow \quad c_1 = 5$$

$$x_1 = 18 \quad \Rightarrow \quad 6c_1 + 6c_2 = 18 \quad \Rightarrow \quad c_1 + c_2 = 3$$

$$c_1 = 5 \quad \Rightarrow \quad c_2 = 3 - 5 = -2$$

The solution of the difference equation is:

$$x_n = 5 \cdot 6^n - 2n \cdot 6^n$$

$$x_n = \big(5 - 2n\big)6^n$$

Example 24–10: Find the solution of the difference equation:

$$4x_{n+2} - 16x_{n+1} + 7x_n = 0, \quad x_0 = 14, \quad x_1 = 49.$$

Solution: Let's insert $x_n = r^n$ in the equation:

$$4r^{n+2} - 16r^{n+1} + 7r^n = 0$$

$$4r^2 - 16r + 7r = 0$$

$$(2r-1)(2r-7) = 0$$

$r = \frac{1}{2}$ or $r = \frac{7}{2}$. We have two distinct roots.

$$x_n = c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{7}{2}\right)^n$$

$$x_0 = 14 \quad \Rightarrow \quad c_1 + c_2 = 14$$

$$x_1 = 49 \quad \Rightarrow \quad \frac{c_1}{2} + \frac{7c_2}{2} = 49$$

Multiply the second equation by 2:

$$\begin{aligned} c_1 + 7c_2 &= 98 \\ c_1 + c_2 &= 14 \end{aligned}$$

Subtracting these, we obtain $6c_2 = 84 \quad \Rightarrow \quad c_2 = 14.$ Inserting this in the equation $c_1 + c_2 = 14$, we see that $c_1 = 0$.

Therefore the solution is $x_n = 14\left(\frac{7}{2}\right)^n$, or equivalently:

$$x_n = \frac{7^{n+1}}{2^{n-1}}$$

EXERCISES

Find the solution of the following first order difference equations:

24–1) $x_{n+1} = 8x_n, \quad x_0 = 1.$

24–2) $x_{n+1} = -2x_n, \quad x_0 = 17.$

24–3) $x_{n+1} = \frac{3}{4}x_n, \quad x_0 = 3.$

24–4) $x_{n+1} = 4x_n + 9, \quad x_0 = 7.$

24–5) $x_{n+1} = 2x_n - 6, \quad x_0 = 6.$

24–6) $x_{n+1} = 5x_n + 1, \quad x_0 = 11.$

24–7) $x_{n+1} = x_n + 1, \quad x_0 = 5.$

24–8) $x_{n+1} = x_n + 9, \quad x_0 = 8.$

24–9) $x_{n+1} = \frac{1}{2}x_n + \frac{3}{2}, \quad x_0 = 7.$

24–10) $x_{n+1} = \frac{3}{5}x_n + 24, \quad x_0 = 90.$

24–11) $x_{n+1} = x_n + \frac{1}{5}, \quad x_0 = 0.$

24–12) $x_{n+1} = -3x_n + 36, \quad x_0 = 10.$

Find the solution of the following second order difference equations:

24–13) $x_{n+2} - 5x_{n+1} + 4x_n = 0.$

24–14) $x_{n+2} + 7x_{n+1} - 18x_n = 0.$

24–15) $x_{n+2} - 25x_n = 0.$

24–16) $x_{n+2} - 6x_{n+1} + 9x_n = 0.$

24–17) $x_{n+2} - 10x_{n+1} + 21x_n = 0, \quad x_0 = 4, \quad x_1 = 8.$

24–18) $x_{n+2} - 6x_{n+1} + 8x_n = 0, \quad x_0 = 9, \quad x_1 = 14.$

24–19) $x_{n+2} + 8x_{n+1} + 16x_n = 0, \quad x_0 = 1, \quad x_1 = -12.$

24–20) $x_{n+2} - 20x_{n+1} + 100x_n = 0, \quad x_0 = 0, \quad x_1 = 100.$

24–21) $6x_{n+2} - 5x_{n+1} + x_n = 0, \quad x_0 = 12, \quad x_1 = 6.$

24–22) $x_{n+2} - 4x_{n+1} + 4x_n = 0, \quad x_0 = 8, \quad x_1 = 16.$

24–23) $16x_{n+2} - 16x_{n+1} + 3x_n = 0, \quad x_0 = 0, \quad x_1 = \frac{1}{2}.$

24–24) $25x_{n+2} - 20x_{n+1} + 4x_n = 0, \quad x_0 = 13, \quad x_1 = 2.$

ANSWERS

24–1) $x_n = 8^n$

24–2) $x_n = 17 \cdot (-2)^n$

24–3) $x_n = \dfrac{3^{n+1}}{4^n}$

24–4) $x_n = 10 \cdot 4^n - 3$

24–5) $x_n = 6$

24–6) $x_n = \dfrac{9}{4} \cdot 5^{n+1} - \dfrac{1}{4}$

24–7) $x_n = 5 + n$

24–8) $x_n = 8 + 9n$

24–9) $x_n = \dfrac{1}{2^{n-2}} + 3$

24–10) $x_n = 30\left(\dfrac{3}{5}\right)^n + 60$

24–11) $x_n = \dfrac{n}{5}$

24–12) $x_n = (-3)^n + 9$

24–13) $x_n = c_1 + c_2 4^n$

24–14) $x_n = c_1 2^n + c_2(-9)^n$

24–15) $x_n = c_1 5^n + c_2(-5)^n$

24–16) $x_n = c_1 3^n + c_2 n 3^n$

24–17) $x_n = 5 \cdot 3^n - 7^n$

24–18) $x_n = -2 \cdot 4^n + 11 \cdot 2^n$

24–19) $x_n = (-4)^n\big(1 + 2n\big)$

24–20) $x_n = n 10^{n+1}$

24–21) $x_n = \dfrac{12}{2^n}$

24–22) $x_n = 2^{n+3}$

24–23) $x_n = \dfrac{3^n - 1}{4^n}$

24–24) $x_n = \left(\dfrac{2}{5}\right)^n\big(13 - 8n\big)$

Chapter 25

Geometric Series

An infinite series is an infinite sum of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

For example,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

Although we are adding an infinite number of positive terms, the sum can be finite. For example, consider the series

$$\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = 1 + 0.1 + 0.01 + 0.001 + \cdots$$

The sum is:

$$1.111111\ldots = 1.\bar{1} = \frac{10}{9}$$

Partial Sum: n^{th} partial sum of a series is the sum of its first n terms:

$$S_n = \sum_{k=1}^{n} a_k$$

We say that the infinite series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges with sum S if the limit

$$S = \lim_{n\to\infty} S_n$$

exists and is finite. Otherwise we say the series diverges.

We can add two convergent series, or multiply a convergent series by a constant to obtain another series which is also convergent.

Similarly, adding or deleting a finite number of terms does not change convergence. For example, the series

$$\sum_{n=0}^{\infty} a_n \quad \text{and} \quad \sum_{n=5}^{\infty} a_n$$

are either both convergent or both divergent.

Geometric Series: The series

$$\sum_{n=0}^{\infty} r^n$$

is called a geometric series. Its partial sum is:

$$S_n = 1 + r + r^2 + r^3 + \cdots + r^n$$

We can find S_n using the following trick:

$$\begin{aligned} rS_n &= r + r^2 + r^3 + r^4 + \cdots + r^{n+1} \\ S_n - rS_n &= (1 + r + \cdots + r^n) - (r + r^2 + \cdots + r^{n+1}) \\ &= 1 - r^{n+1} \end{aligned}$$

In other words:

$$S_n = \frac{1 - r^{n+1}}{1 - r}$$

What happens as $n \to \infty$? Is the series convergent?

Clearly, the geometric series is convergent if $|r| < 1$ and its sum is:

$$S = \lim_{n\to\infty} S_n = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

For example,

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

is a convergent series and its sum is:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2$$

If $|r| \geqslant 1$ then the geometric series is divergent. For example

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \cdots$$

is a divergent series.

Example 25–1: Find the sum of the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$$

Solution: Here $r = \dfrac{1}{3}$. The series is convergent because $\dfrac{1}{3} < 1$.

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

Example 25–2: Find the sum of the geometric series

$$1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \cdots$$

Solution: Here $r = -\dfrac{3}{4} \quad \Rightarrow \quad |r| < 1$ therefore the series is convergent.

$$\sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n = \frac{1}{1 - \left(-\frac{3}{4}\right)} = \frac{1}{\frac{7}{4}} = \frac{4}{7}$$

Example 25–3: Find the sum of the geometric series

$$\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\cdots$$

Solution: Here $r=\frac{1}{5}$. The series is convergent because

$$\left|\frac{1}{5}\right|<1$$

But we have to be careful, the series is not in standard form. Therefore the answer is NOT $\frac{1}{1-\frac{1}{5}}$.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{5^n} &= \frac{1}{1-\frac{1}{5}}-1\\ &= \frac{5}{4}-1\\ &= \frac{1}{4}\end{aligned}$$

An alternative method is:

$$\begin{aligned}\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\cdots &= \frac{1}{5}\left(1+\frac{1}{5}+\frac{1}{25}+\cdots\right)\\ &= \frac{1}{5}\cdot\frac{1}{1-\frac{1}{5}}\\ &= \frac{1}{5}\cdot\frac{5}{4}\\ &= \frac{1}{4}\end{aligned}$$

Example 25–4: Find the sum of the series

$$\sum_{n=2}^{\infty}\frac{2^{n-1}}{3^{2n}}$$

if it is convergent.

Solution: Here $r=\frac{2}{3^2}=\frac{2}{9}$ therefore the series is convergent.

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{2^{n-1}}{3^{2n}} &= \sum_{n=2}^{\infty}\frac{2^{n-1}}{9^n}\\ &= \sum_{n=0}^{\infty}\frac{2^{n+1}}{9^{n+2}}\\ &= \frac{2}{9^2}\sum_{n=0}^{\infty}\left(\frac{2}{9}\right)^n\\ &= \frac{2}{81}\cdot\frac{1}{1-\frac{2}{9}}\\ &= \frac{2}{81}\cdot\frac{9}{7}\\ &= \frac{2}{63}\end{aligned}$$

Example 25–5: Is the series $\sum_{n=0}^{\infty} e^n$ convergent?

Solution: This is a geometric series with $r = e = 2.717\ldots$.

$r > 1$ therefore the series is divergent.

Example 25–6: Is the series $\sum_{n=0}^{\infty} e^{-n}$ convergent?

Solution: This is a geometric series with $r = e^{-1} = \frac{1}{e}$.

$r < 1$ therefore the series is convergent.

Its sum is:
$$\frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}$$

Example 25–7: Is the series
$$\frac{5}{2} - \frac{25}{4} + \frac{125}{8} - \cdots$$
convergent?

Solution: This is a geometric series with $r = -\frac{5}{2}$.

$|r| > 1$ therefore the series is divergent.

Example 25–8: Find the sum of the series
$$\frac{1}{7} - \frac{6}{7^2} + \frac{6^2}{7^3} - \frac{6^3}{7^4} + \cdots$$
(if it is convergent)

Solution: This is a geometric series with $r = -\frac{6}{7}$.

$|r| < 1$ therefore the series is convergent.

$$\begin{aligned}
\frac{1}{7} - \frac{6}{7^2} + \frac{6^2}{7^3} - \frac{6^3}{7^4} + \cdots &= \frac{1}{7}\left(1 - \frac{6}{7} + \frac{6^2}{7^2} - \frac{6^3}{7^2} + \cdots\right) \\
&= \frac{1}{7}\sum_{n=0}^{\infty} (-1)^n \frac{6^n}{7^n} \\
&= \frac{1}{7}\sum_{n=0}^{\infty} \left(-\frac{6}{7}\right)^n \\
&= \frac{1}{7} \cdot \frac{1}{1-(-\frac{6}{7})} \\
&= \frac{1}{7} \cdot \frac{1}{\frac{13}{7}} \\
&= \frac{1}{7} \cdot \frac{7}{13} \\
&= \frac{1}{13}
\end{aligned}$$

EXERCISES

Find the sum of the given series if it is convergent:

25–1) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$

25–2) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{7}{3}\right)^n$

25–3) $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{4}{5}\right)^n$

25–4) $\displaystyle\sum_{n=0}^{\infty}\frac{12}{3^{n+1}}$

25–5) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}$

25–6) $\displaystyle\sum_{n=0}^{\infty}(e-1)^n$

25–7) $1+0.6+0.36+0.216+0.1296+\cdots$

25–8) $\displaystyle\sum_{n=0}^{\infty}e^{\frac{n}{2}}$

25–9) $\displaystyle\sum_{n=0}^{\infty}e^{-\frac{n}{2}}$

25–10) $\displaystyle\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}+\cdots$

Find the sum of the given series if it is convergent:

25–11) $\displaystyle\sum_{n=1}^{\infty}\frac{4^{n+1}}{5^n}$

25–12) $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{12^{n-1}}$

25–13) $\displaystyle\sum_{n=1}^{\infty}\frac{5^{2n-1}}{3^{3n+2}}$

25–14) $\displaystyle\sum_{n=1}^{\infty}\frac{3^{3n-1}}{4^{2n+1}}$

25–15) $\displaystyle\sum_{n=2}^{\infty}\frac{2^{2n+1}}{5^{n-1}}$

25–16) $\displaystyle\sum_{n=2}^{\infty}\frac{(-8)^n}{2^{3n+1}}$

25–17) $\displaystyle\sum_{n=2}^{\infty}\frac{(-3)^{2n-1}}{10^{n-2}}$

25–18) $\displaystyle 4+6+9+\frac{27}{2}+\frac{81}{4}+\cdots$

25–19) $\displaystyle\frac{12}{5}+\frac{24}{25}+\frac{48}{125}+\frac{96}{625}+\cdots$

25–20) $\displaystyle 2\cdot 9+2\cdot\frac{9^2}{11}+2\cdot\frac{9^3}{11^2}+2\cdot\frac{9^4}{11^3}+\cdots$

ANSWERS

25–1) 3

25–2) Divergent

25–3) $\frac{5}{9}$

25–4) 6

25–5) $\frac{4}{5}$

25–6) Divergent

25–7) 2.5

25–8) Divergent

25–9) $\frac{\sqrt{e}}{\sqrt{e}-1}$

25–10) $\frac{3}{4}$

25–11) 16

25–12) $\frac{24}{7}$

25–13) $\frac{5}{18}$

25–14) Divergent

25–15) 32

25–16) Divergent

25–17) -270

25–18) Divergent

25–19) 4

25–20) 99

Chapter 26

Matrices and Basic Operations

An $n \times m$ matrix is a rectangular array of numbers.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

This matrix has n **rows** and m **columns**. The numbers a_{ij} are called **entries**.

If $m = n$, it is called a **square matrix**, for example

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 9 & 8 & 7 \end{bmatrix}$$

is a square matrix. The **main diagonal** entries of a square matrix are the entries $a_{11}, a_{22}, \ldots, a_{nn}$.

$$A = \begin{bmatrix} \boxed{1} & 3 & 5 \\ 0 & \boxed{2} & 4 \\ 9 & 8 & \boxed{7} \end{bmatrix}$$

A **diagonal matrix** is a square matrix where all the entries that are not on the main diagonal are zero, for example

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

An **upper triangular** matrix is a square matrix where all elements below diagonal are zero, and a **lower triangular** matrix is a square matrix where all elements above diagonal are zero, for example:

$$C = \begin{bmatrix} 6 & 4 & 2 & 11 \\ 0 & 2 & 18 & 20 \\ 0 & 0 & 9 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 15 & 7 & 0 & 0 \\ 6 & 33 & 8 & 0 \\ -5 & 13 & 40 & 1 \end{bmatrix}$$

The **transpose** of a matrix A is A^T, obtained by interchanging rows and columns, for example:

$$E = \begin{bmatrix} 3 & 5 & 7 \\ 10 & 2 & 4 \\ -8 & 0 & 9 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 3 & 10 & -8 \\ 5 & 2 & 0 \\ 7 & 4 & 9 \end{bmatrix}$$

are transposes of each other. $F = E^T$ and $E = F^T$

If $A = A^T$, we call it a **symmetric matrix**, for example:

$$A = \begin{bmatrix} 1 & 2 & 9 \\ 2 & 4 & 7 \\ 9 & 7 & 5 \end{bmatrix}$$

We can add and subtract matrices by adding or subtracting their corresponding entries, for example:

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 8 & 22 \end{bmatrix} + \begin{bmatrix} 12 & -3 & 5 \\ 9 & 1 & 11 \end{bmatrix} = \begin{bmatrix} 13 & -3 & 3 \\ 11 & 9 & 33 \end{bmatrix}$$

If matrix dimensions are different, we can not add or subtract them, for example:

$$\begin{bmatrix} 7 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 6 & 9 \\ 1 & 2 & 3 \\ 5 & 4 & 0 \end{bmatrix} = ??$$

is undefined.

If A is a matrix and c is a scalar (a number), then $B = cA$ is the **scalar multiplication** of c and A.

$$b_{ij} = c\, a_{ij}$$

In other words, we multiply each entry. For example:

$$7 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 20 & -1 \\ 4 & 5 & 11 \end{bmatrix} = \begin{bmatrix} 7 & 14 & 21 \\ 0 & 140 & -7 \\ 28 & 35 & 77 \end{bmatrix}$$

Example 26–1: Let

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 2 \\ 11 & 9 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad D = \begin{bmatrix} 8 & 3 \\ 4 & 1 \\ 2 & 0 \end{bmatrix}$$

Calculate the following if possible:

a) $3A + B$

b) $A - B^T$

c) $A + C$

d) $C - D$

e) $2C + D^T$

Solution: a) $3A + B = \begin{bmatrix} 0 & 14 \\ 11 & 24 \end{bmatrix}$

b) $A - B^T = \begin{bmatrix} 4 & -7 \\ -2 & -4 \end{bmatrix}$

c) $A + C$ is undefined, dimensions are different.

d) $C - D$ is undefined, dimensions are different.

e) $2C + D^T = \begin{bmatrix} 10 & 10 & 12 \\ 7 & 9 & 12 \end{bmatrix}$

Matrix Multiplication: If A is an $n \times k$ matrix and B is a $k \times m$ matrix, then $C = AB$ is an $n \times m$ matrix. The entries of C are calculated using:

$$c_{ij} = \sum_{p=1}^{k} a_{ip} b_{pj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

For example

$$\begin{bmatrix} 3 & 4 & 8 \\ 1 & 7 & 5 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 10 & 7 \end{bmatrix}$$

is impossible, because number of columns of the first matrix and the number of rows of the second matrix are not equal.

$$AB = \begin{bmatrix} 3 & 4 & 8 \\ 1 & 7 & 5 \end{bmatrix} \begin{bmatrix} 5 & 2 & 7 & 14 \\ 10 & 8 & 21 & 1 \\ 0 & -3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix}$$

This is possible because A has three columns and B has 3 rows. Some of the entries are:

$$c_{11} = 3 \cdot 5 + 4 \cdot 10 + 8 \cdot 0 = 55$$

$$c_{12} = 3 \cdot 2 + 4 \cdot 8 + 8 \cdot -3 = 14$$

$$c_{21} = 1 \cdot 5 + 7 \cdot 10 + 5 \cdot 0 = 75$$

etc. To find c_{ij}, multiply the i^{th} row of A with j^{th} column of B. For example, to find c_{13}:

$$\begin{bmatrix} \boxed{\bullet} & \boxed{\bullet} & \boxed{\bullet} \\ \bullet & \bullet & \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \boxed{\bullet} & \bullet \\ \bullet & \bullet & \boxed{\bullet} & \bullet \\ \bullet & \bullet & \boxed{\bullet} & \bullet \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \boxed{\bullet} & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

Example 26–2: Let $A = \begin{bmatrix} 1 & 7 \\ 8 & 2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 3 & 5 \\ 9 & 0 \end{bmatrix}$,

$$C = \begin{bmatrix} 1 & 6 & 5 \\ 0 & 4 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 7 & -2 & 3 \\ -1 & 2 & 4 \end{bmatrix}.$$

Find the following matrix products (if possible).

a) AB
b) BA
c) CD
d) CD^T
e) $C^T D$
f) DB

Solution: a) $\begin{bmatrix} 1 & 7 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 9 & 0 \end{bmatrix} = \begin{bmatrix} 66 & 5 \\ 42 & 40 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 43 & 31 \\ 9 & 63 \end{bmatrix}$

c) Undefined

d) $\begin{bmatrix} 1 & 6 & 5 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ -2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 31 \\ -2 & 16 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 \\ 6 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 7 & -2 & 3 \\ -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -2 & 3 \\ 38 & -4 & 34 \\ 33 & -6 & 23 \end{bmatrix}$

f) Undefined

Matrix product is associative:

$$A\big(BC\big) = \big(AB\big)C$$

For example, consider the product

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 7 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ -4 & 4 \end{bmatrix}$$

We can find the result in two different ways:

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \left(\begin{bmatrix} 3 & -1 \\ 7 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ -4 & 4 \end{bmatrix} \right) &= \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 19 & 14 \\ 35 & 42 \end{bmatrix} \\ &= \begin{bmatrix} 89 & 98 \\ 140 & 168 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 7 & 0 \end{bmatrix} \right) \begin{bmatrix} 5 & 6 \\ -4 & 4 \end{bmatrix} &= \begin{bmatrix} 17 & -1 \\ 28 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ -4 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 89 & 98 \\ 140 & 168 \end{bmatrix} \end{aligned}$$

The results are the same.

But matrix product is not commutative:

$$AB \neq BA \quad \text{(In general)}$$

The transpose of the product is product of transposes in reverse order:

$$\big(AB\big)^T = B^T A^T$$

An **identity matrix** is a diagonal matrix whose all nonzero entries are 1. We denote identity matrices by I_n or simply I.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The product of an identity matrix with any other matrix A gives the result A. (when the product is defined.) For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 18 & 6 & 9 \\ 5 & 7 & 11 \end{bmatrix} = \begin{bmatrix} 18 & 6 & 9 \\ 5 & 7 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 18 & 6 & 9 \\ 5 & 7 & 11 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 6 & 9 \\ 5 & 7 & 11 \end{bmatrix}$$

For a square matrix A, we have:

- $A^n = AA \cdots A \quad (n \text{ times})$
- $A^n A^m = A^{n+m}$
- $\big(A^n\big)^m = A^{nm}$
- $A^0 = I$

The powers of a diagonal matrix are especially easy to calculate, for example:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}^8 = \begin{bmatrix} 3^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

EXERCISES

The matrices A, B, C, D are defined as:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 1 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 1 & -5 \\ -1 & -2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 1 \\ 0 & 6 \\ 3 & 0 \end{bmatrix}.$$

Perform the following operations if possible:

26–1) $A + 4B$

26–2) AB

26–3) CD

26–4) DC

26–5) $C + D$

26–6) ADC

26–7) DAC

26–8) $2BA - 5DC$

26–9) $CB - AD$

26–10) $AD - BD$

26–11) $A + B^T$

26–12) $(A + B)^T$

26–13) $(A^T - 2B)^T$

26–14) $C^T + D$

26–15) $D^T - 2C^T$

26–16) $B^2 - 2B - I$

Let $A = \begin{bmatrix} 2 & -1 \\ 5 & 7 \end{bmatrix}$,

$B = \begin{bmatrix} 3 & 4 & 0 \\ 8 & 5 & 7 \end{bmatrix}$,

$C = \begin{bmatrix} -2 & 9 & 1 \\ 4 & 6 & -5 \\ 3 & 2 & 0 \end{bmatrix}$,

$D = \begin{bmatrix} 6 & 2 \\ 0 & -2 \\ 5 & 1 \end{bmatrix}$.

Calculate the following if possible:

26–17) AB^T

26–18) BC

26–19) DC

26–20) $(BD)^T$

26–21) DB

26–22) CD

Evaluate the following matrix products if they are defined:

26–23) $\begin{bmatrix} 1 & 3 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 10 & 7 & 9 \\ 0 & 1 & 4 \end{bmatrix}$

26–24) $\begin{bmatrix} 10 & 7 & 9 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 9 \end{bmatrix}$

26–25) $\begin{bmatrix} 2 & 0 & 3 & -1 \\ 8 & 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 2 & 2 \\ 6 & 5 \\ 9 & -2 \end{bmatrix}$

26–26) $\begin{bmatrix} 7 & 0 \\ 2 & 2 \\ 6 & 5 \\ 9 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 & -1 \\ 8 & 5 & 0 & 4 \end{bmatrix}$

26–27) $\begin{bmatrix} 4 & -3 & 8 \\ 9 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 5 & 7 \\ 2 & 4 & 12 \\ 0 & 8 & 0 \end{bmatrix}$

26–28) $\begin{bmatrix} 9 & 5 & 1 \\ 4 & 0 & 8 \\ 7 & 2 & 11 \end{bmatrix} \begin{bmatrix} 3 & 0 & -3 \\ 6 & 8 & 1 \\ -4 & 2 & -1 \end{bmatrix}$

26–29) $\begin{bmatrix} 3 & 0 & -3 \\ 6 & 8 & 1 \\ -4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 9 & 5 & 1 \\ 4 & 0 & 8 \\ 7 & 2 & 11 \end{bmatrix}$

26–30) $\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 8 & 10 & 12 & 16 \\ 2 & 3 & 5 & 3 \end{bmatrix}$

ANSWERS

26–1) $\begin{bmatrix} 33 & 6 & 0 \\ 11 & 3 & 6 \\ 4 & 13 & 9 \end{bmatrix}$

26–2) $\begin{bmatrix} 14 & 1 & 2 \\ 1 & 3 & 11 \\ 47 & -10 & -23 \end{bmatrix}$

26–3) $\begin{bmatrix} -7 & 8 \\ -4 & -13 \end{bmatrix}$

26–4) $\begin{bmatrix} 7 & 2 & -20 \\ -6 & -12 & 0 \\ 6 & 3 & -15 \end{bmatrix}$

26–5) Undefined

26–6) $\begin{bmatrix} -5 & -22 & -20 \\ -13 & -32 & -10 \\ -44 & -73 & 25 \end{bmatrix}$

26–7) Undefined

26–8) $\begin{bmatrix} -21 & 28 & 104 \\ 44 & 82 & -14 \\ -2 & 37 & 27 \end{bmatrix}$

26–9) Undefined

26–10) $\begin{bmatrix} -28 & -1 \\ -13 & 14 \\ -17 & 22 \end{bmatrix}$

26–11) $\begin{bmatrix} 9 & 5 & 0 \\ 0 & 3 & 4 \\ 4 & 6 & -3 \end{bmatrix}$

26–12) $\begin{bmatrix} 9 & 2 & 4 \\ 3 & 3 & 7 \\ 0 & 3 & -3 \end{bmatrix}$

26–13) $\begin{bmatrix} -15 & -4 & 0 \\ -3 & 3 & -2 \\ 4 & 3 & -15 \end{bmatrix}$

26–14) $\begin{bmatrix} 6 & 0 \\ 1 & 4 \\ -2 & 0 \end{bmatrix}$

26–15) Undefined

26–16) $\begin{bmatrix} 50 & 6 & 1 \\ 18 & 4 & 2 \\ 6 & 4 & 9 \end{bmatrix}$

26–17) Undefined

26–18) $\begin{bmatrix} 10 & 51 & -17 \\ 25 & 116 & -17 \end{bmatrix}$

26–19) Undefined

26–20) $\begin{bmatrix} 18 & 83 \\ -2 & 13 \end{bmatrix}$

26–21) $\begin{bmatrix} 34 & 34 & 14 \\ -16 & -10 & -14 \\ 23 & 25 & 7 \end{bmatrix}$

26–22) $\begin{bmatrix} -7 & -21 \\ -1 & -9 \\ 18 & 2 \end{bmatrix}$

26–23) $\begin{bmatrix} 10 & 10 & 21 \\ 40 & 37 & 72 \end{bmatrix}$

26–24) Undefined.

26–25) $\begin{bmatrix} 23 & 17 \\ 102 & 2 \end{bmatrix}$

26–26) $\begin{bmatrix} 14 & 0 & 21 & -7 \\ 20 & 10 & 6 & 6 \\ 52 & 25 & 18 & 14 \\ 2 & -10 & 27 & -17 \end{bmatrix}$

26–27) $\begin{bmatrix} -2 & 72 & -8 \\ 9 & 69 & 63 \end{bmatrix}$

26–28) $\begin{bmatrix} 53 & 42 & -23 \\ -20 & 16 & -20 \\ -11 & 38 & -30 \end{bmatrix}$

26–29) $\begin{bmatrix} 6 & 9 & -30 \\ 93 & 32 & 81 \\ -35 & -22 & 1 \end{bmatrix}$

26–30) $\begin{bmatrix} 14 & 17 & 19 & 29 \\ 0 & 2 & 8 & -4 \end{bmatrix}$

Chapter 27

Row Reduction

Row-Echelon Form:
A matrix that satisfies the following conditions is in **row-echelon form**:

- If there are rows consisting of zeros only, they are at the bottom.
- The first nonzero item (from left to right) of each row is 1. It is called a **leading 1**.
- Each leading 1 is to the right of the other leading 1's above it.

If a matrix is in row-echelon form, we can solve the system of equations represented by that matrix easily.
Some examples of matrices in row-echelon form (REF) are:

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{1} & 4 \\ 0 & 0 & \boxed{1} \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} & -5 & 10 & -1 \\ 0 & \boxed{1} & 9 & 7 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 5 & 9 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} & 6 & 12 & 4 \\ 0 & \boxed{1} & 3 & 5 \\ 0 & 0 & \boxed{1} & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} & 23 & 0 & -8 \\ 0 & \boxed{1} & 12 & 7 \\ 0 & 0 & \boxed{1} & 9 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

The following matrices are NOT in REF:

$$\begin{bmatrix} 1 & 5 & 11 & 8 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 44 & 0 & 16 \\ 0 & 1 & 21 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -3 & 2 & 6 & 12 \\ 0 & 1 & 48 & 17 & 21 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 1 & 33 & -7 \end{bmatrix}$$

Elementary Row Operations:
There are three types of elementary row operations. We can:

- Interchange two rows,
- Multiply a row by a nonzero constant,
- Add a multiple of a row to another row.

If we do these operations on a matrix, we obtain a **row-equivalent** matrix.

Note that these operations are similar to operations on linear systems of equations, in the sense that they do not change the solution set.

We can reduce matrices to REF using elementary row operations in a systematic way.

It is possible to use different steps to reduce a matrix. For example, if you have a 4 at the top left and 1 below it, you can multiply row 1 by $\frac{1}{4}$ or you can interchange row 1 and row 2.

Example 27–1: Find a REF matrix that is row equivalent to

$$\begin{bmatrix} 0 & 1 & 4 \\ 2 & 4 & -6 \\ 3 & 8 & 0 \end{bmatrix}$$

Solution: Interchange the first two rows:

$$R_1 \longleftrightarrow R_2 \qquad \Longrightarrow \begin{bmatrix} 2 & 4 & -6 \\ 0 & 1 & 4 \\ 3 & 8 & 0 \end{bmatrix}$$

Divide row 1 by two:

$$R_1 \rightarrow \frac{1}{2} R_1 \qquad \Longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 3 & 8 & 0 \end{bmatrix}$$

Multiply row 1 by three and subtract from row 3:

$$R_3 \rightarrow R_3 - 3R_1 \Longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 0 & 2 & 9 \end{bmatrix}$$

Multiply row 2 by two and subtract from row 3:

$$R_3 \rightarrow R_3 - 2R_2 \Longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Reduced Row-Echelon Form:
Consider a matrix that is in row-echelon form. If it satisfies the further condition

- If there's a leading 1 in a column, all other entries in that column are zero.

we say that matrix is in **reduced row-echelon form** (RREF). An identity matrix is a typical RREF matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Some other examples are:

$$\begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & 24 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 15 & 0 & 0 & 23 & 36 \\ 0 & 0 & 1 & 0 & 40 & 7 \\ 0 & 0 & 0 & 1 & 54 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 27–2: Using elementary row operations, reduce the following matrix to RREF:

$$\begin{bmatrix} 2 & -6 & -2 & 8 \\ -2 & 11 & 7 & 12 \\ -3 & 10 & 5 & -1 \end{bmatrix}$$

Solution: $R_1 \to \dfrac{1}{2} R_1$ $$\begin{bmatrix} 1 & -3 & -1 & 4 \\ -2 & 11 & 7 & 12 \\ -3 & 10 & 5 & -1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 + 3R_1$ $$\begin{bmatrix} 1 & -3 & -1 & 4 \\ 0 & 5 & 5 & 20 \\ 0 & 1 & 2 & 11 \end{bmatrix}$$

$R_2 \to \dfrac{1}{5} R_2$ $$\begin{bmatrix} 1 & -3 & -1 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 2 & 11 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ $$\begin{bmatrix} 1 & -3 & -1 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

This is REF.

$R_2 \to R_2 - R_3$
$R_1 \to R_1 + R_3$ $$\begin{bmatrix} 1 & -3 & 0 & 11 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$R_1 \to R_1 + 3R_2$ $$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

This is RREF.

Method of Reduction:

- A leading 1 in first in position a_{11} if possible. If not, next right position (a_{12}).
- All entries below the leading 1 must be zero.
- Similarly, a leading 1 in second row in position a_{22} if possible. If not, next right position (a_{23}).
- All entries below the leading 1 must be zero.
- $\cdots$

At the end of this, we obtain a REF matrix.

If we want a RREF matrix, we should continue as follows:

- All entries above the lowest (rightmost) leading 1 must be zero.
- All entries above the next leading 1 must be zero.
- $\cdots$

We can illustrate this procedure on a sample 4×4 matrix as follows:

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \to \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \to \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \end{bmatrix} \to \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \end{bmatrix} \to \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & 1 & \bullet \\ 0 & 0 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & \bullet \end{bmatrix} \to \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix we obtained at the end of this procedure is in REF. Now in the second part, we reduce the matrix further into RREF:

$$\begin{bmatrix} 1 & \bullet & \bullet & 0 \\ 0 & 1 & \bullet & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \bullet & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EXERCISES

Are the following matrices in row-echelon form (REF)?

27–1) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

27–2) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

27–3) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

27–4) $\begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

27–5) $\begin{bmatrix} 1 & -5 & 3 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{bmatrix}$

27–6) $\begin{bmatrix} 1 & 4 & -8 & 40 \\ 0 & 1 & 5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

27–7) $\begin{bmatrix} 1 & 4 & -8 & 2 & 0 \\ 0 & 1 & 5 & -7 & 6 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

Are the following matrices in reduced row-echelon form (RREF)?

27–8) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

27–9) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

27–10) $\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

27–11) $\begin{bmatrix} 1 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$

27–12) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

27–13) $\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

27–14) $\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 1 & 0 & 8 & 3 \end{bmatrix}$

Using elementary row operations, reduce the following matrices to REF.

27–15) $\begin{bmatrix} 1 & 4 \\ 2 & 12 \end{bmatrix}$

27–16) $\begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 4 \\ 2 & 7 & 13 \end{bmatrix}$

27–17) $\begin{bmatrix} 3 & 0 & 6 & -6 \\ 0 & 1 & 0 & -3 \\ 2 & -3 & 4 & 5 \end{bmatrix}$

27–18) $\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 2 & 0 & 1 \\ 1 & 4 & 1 & 4 \\ -4 & 1 & 3 & 4 \end{bmatrix}$

27–19) $\begin{bmatrix} 3 & 21 \\ 2 & 18 \end{bmatrix}$

27–20) $\begin{bmatrix} 4 & -8 & 16 \\ -8 & 15 & -44 \\ -3 & 7 & -4 \end{bmatrix}$

27–21) $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 6 & 3 & 3 & 9 \\ 8 & 5 & 9 & 21 \end{bmatrix}$

27–22) $\begin{bmatrix} 2 & 3 \\ 1 & 7 \\ 5 & -6 \end{bmatrix}$

Using elementary row operations, reduce the following matrices to RREF.

27–23) $\begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 5 \end{bmatrix}$

27–24) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 7 & 7 & 4 \end{bmatrix}$

27–25) $\begin{bmatrix} 2 & 10 & 5 & -1 \\ 0 & 0 & 1 & 4 \\ 1 & 5 & 3 & 2 \end{bmatrix}$

27–26) $\begin{bmatrix} 1 & 1 & 2 & -2 \\ 0 & 1 & 3 & -4 \\ 1 & 2 & 5 & -6 \\ 2 & 1 & 1 & 0 \end{bmatrix}$

27–27) $\begin{bmatrix} 3 & -6 & 12 \\ -1 & 14 & 92 \\ 1 & 3 & 51 \end{bmatrix}$

27–28) $\begin{bmatrix} 1 & 2 & -2 & 6 & -28 \\ 4 & 7 & -7 & 22 & -98 \\ 2 & 9 & -6 & 25 & -111 \end{bmatrix}$

27–29) $\begin{bmatrix} 1 & 2 & 16 \\ 4 & 7 & 50 \end{bmatrix}$

27–30) $\begin{bmatrix} 5 & 20 & 75 \\ 2 & 16 & 46 \\ 3 & 11 & 43 \end{bmatrix}$

ANSWERS

27–1) Yes

27–2) Yes

27–3) No

27–4) No

27–5) Yes

27–6) No

27–7) No

27–8) Yes

27–9) Yes

27–10) Yes

27–11) No

27–12) Yes

27–13) Yes

27–14) No

You may find slightly different results for the exercises on this page. REF form of a matrix depends on the operations we choose, but RREF form is unique.

27–15) $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

27–16) $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

27–17) $\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

27–18) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

27–19) $\begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}$

27–20) $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{bmatrix}$

27–21) $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

27–22) $\begin{bmatrix} 1 & 7 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

27–23) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

27–24) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

27–25) $\begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

27–26) $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

27–27) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

27–28) $\begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & -9 \\ 0 & 0 & 1 & 1 & 5 \end{bmatrix}$

27–29) $\begin{bmatrix} 1 & 0 & -12 \\ 0 & 1 & 14 \end{bmatrix}$

27–30) $\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Chapter 28

Systems of Linear Equations

Equations and Solutions: A system of n linear equations in m unknowns is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n \end{aligned}$$

For example,

$$\begin{aligned} 2x_1 - 3x_2 &= 7 \\ x_1 + x_2 &= 6 \end{aligned}$$

is a system of 2 linear equations in 2 unknowns. We can easily find the solution as: $x_1 = 5, \quad x_2 = 1.$

$$\begin{aligned} 2x_1 + 2x_2 &= 10 \\ x_1 + x_2 &= 6 \end{aligned}$$

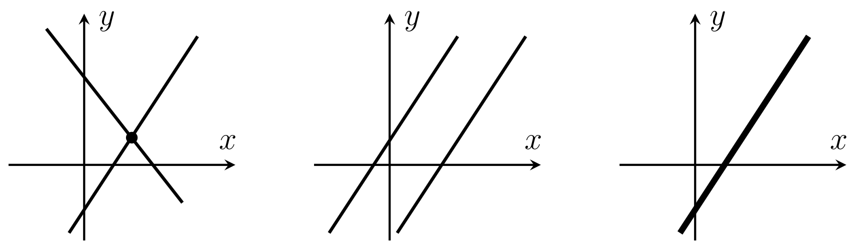
is another system. It has no solution, in other words, it is **inconsistent**. (Can you see why?)

Two important questions about systems of equations are:

- Is there a solution? (Is the system consistent?)
- Supposing there is a solution, are there other solutions or is the solution unique?

A geometric interpretation will help us analyze this problem. The solution of 2 linear equations in 2 unknowns can be considered as the point of intersection of 2 lines in plane. Clearly, there are three possibilities:

- The lines intersect at a single point. (Unique solution)
- The lines are paralel. (No solution)
- The lines are identical. (Infinitely many solutions)



Gaussian Elimination:
The method of Gaussian elimination can be summarized as:

- Represent the system of equations by an augmented matrix.
- Using row operations, obtain row echelon form (REF) of this matrix.
- Use back-substitution to find the unknowns.

The main idea is that, each row represents one equation. So, row operations simplify the system but do not change the solution.

For example, consider:

$$\begin{aligned} x - 3y &= -1 \\ 2x + y &= 12 \end{aligned}$$

The augmented matrix representing this system is:

$$\left[\begin{array}{cc|c} 1 & -3 & -1 \\ 2 & 1 & 12 \end{array}\right]$$

Reduction to REF gives:

$$\left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

The system of equations is:

$$\begin{aligned} x - 3y &= -1 \\ y &= 2 \end{aligned}$$

Back substitution gives:

$$y = 2 \quad \Rightarrow \quad x = 5.$$

The system

$$\begin{aligned} x + 5y &= 6 \\ 2x + 10y &= 15 \end{aligned}$$

has no solution. We can see this using Gaussian elimination as follows:

$$\left[\begin{array}{cc|c} 1 & 5 & 6 \\ 2 & 10 & 15 \end{array}\right]$$

Subtract 2 times the first row from the second row:

$$\left[\begin{array}{cc|c} 1 & 5 & 6 \\ 0 & 0 & 3 \end{array}\right]$$

Back substitution gives:

$$\begin{aligned} 0 &= 3 \\ x + 5y &= 6 \end{aligned}$$

But obviously $0 \neq 3$. We have a contradiction. Therefore there is no solution. The similar system

$$\begin{aligned} x + 5y &= 6 \\ 2x + 10y &= 12 \end{aligned}$$

has infinitely many solutions. Using Gaussian elimination:

$$\left[\begin{array}{cc|c} 1 & 5 & 6 \\ 2 & 10 & 12 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 5 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

Back substitution gives:

$$\begin{aligned} 0 &= 0 \\ x + 5y &= 6 \end{aligned}$$

We can choose y in any way we like. It is a free parameter. So the solution is:

$$x = 6 - 5y$$

Example 28–1: Solve the system of equations

$$\begin{aligned} 2x + 8y + 4z &= 14 \\ 2x + 7y + 3z &= 7 \\ -5x - 18y - 5z &= 15 \end{aligned}$$

using Gaussian elimination.

Solution: First, we will use an augmented matrix to express the system of equations in a simple way:

$$\left[\begin{array}{ccc|c} 2 & 8 & 4 & 14 \\ 2 & 7 & 3 & 7 \\ -5 & -18 & -5 & 15 \end{array}\right]$$

Then we will use row operations to reduce this matrix to REF:

$$R_1 \to \frac{1}{2} R_1 \qquad \left[\begin{array}{ccc|c} 1 & 4 & 2 & 7 \\ 2 & 7 & 3 & 7 \\ -5 & -18 & -5 & 15 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1 \qquad \left[\begin{array}{ccc|c} 1 & 4 & 2 & 7 \\ 0 & -1 & -1 & -7 \\ -5 & -18 & -5 & 15 \end{array}\right]$$

$$R_2 \to -R_2 \qquad \left[\begin{array}{ccc|c} 1 & 4 & 2 & 7 \\ 0 & 1 & 1 & 7 \\ -5 & -18 & -5 & 15 \end{array}\right]$$

$$R_3 \to R_3 + 5R_1 \qquad \left[\begin{array}{ccc|c} 1 & 4 & 2 & 7 \\ 0 & 1 & 1 & 7 \\ 0 & 2 & 5 & 50 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2 \qquad \left[\begin{array}{ccc|c} 1 & 4 & 2 & 7 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 3 & 36 \end{array}\right]$$

$$R_3 \to \frac{1}{3} R_3 \qquad \left[\begin{array}{ccc|c} 1 & 4 & 2 & 7 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 12 \end{array}\right]$$

This matrix is in REF. The system of equations represented by that is:

$$\begin{aligned} x + 4y + 2z &= 7 \\ y + z &= 7 \\ z &= 12 \end{aligned}$$

At this point, we start the back substitution:

$$z = 12$$

$$y + 12 = 7 \quad \Rightarrow \quad y = -5$$

$$x - 20 + 24 = 7 \quad \Rightarrow \quad x = 3$$

Clearly, the solution is unique.

Example 28–2: Solve the following system of equations:

$$\begin{aligned} x + z &= 4 \\ 5x + 4y - 7z &= 16 \\ 2x - 3y + 11z &= 11 \end{aligned}$$

Solution: The augmented matrix is:

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 5 & 4 & -7 & 16 \\ 2 & -3 & 11 & 11 \end{array}\right]$$

Using row operations, we obtain:

$$\begin{array}{l} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 4 & -12 & -4 \\ 0 & -3 & 9 & 3 \end{array}\right]$$

$$R_2 \to \frac{1}{4} R_2 \quad \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & -3 & 9 & 3 \end{array}\right]$$

$$R_3 \to R_3 + 3R_2 \quad \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The back substitution gives:

$$\begin{aligned} y - 3z &= -1 \quad \Rightarrow \quad y = -1 + 3z \\ x + z &= 4 \quad \Rightarrow \quad x = 4 - z \end{aligned}$$

Here, z is a free parameter. There are infinitely many solutions.

Example 28–3: Solve the following system of equations:

$$\begin{aligned} x + 4y + z &= -1 \\ 2x + 10y &= 8 \\ x + 3y + 2z &= -5 \end{aligned}$$

Solution: The augmented matrix is:

$$\left[\begin{array}{rrr|r} 1 & 4 & 1 & -1 \\ 2 & 10 & 0 & 8 \\ 1 & 3 & 2 & -5 \end{array}\right]$$

Using row operations, we obtain:

$$\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 4 & 1 & -1 \\ 0 & 2 & -2 & 10 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$$R_2 \to \frac{1}{2} R_2 \quad \left[\begin{array}{rrr|r} 1 & 4 & 1 & -1 \\ 0 & 1 & -1 & 5 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$$R_3 \to R_3 + R_2 \quad \left[\begin{array}{rrr|r} 1 & 4 & 1 & -1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The back substitution gives:

$$\begin{aligned} 0 &= 1 \\ y - z &= 5 \\ x + 4y + z &= -1 \end{aligned}$$

$0 = 1$ is impossible, a contradiction. Therefore there are no solutions. The system is inconsistent.

Gauss-Jordan Elimination:
Gauss–Jordan elimination is similar to Gaussian elimination, but instead of stopping at REF, we go all the way to RREF. So, the reduction of the augmented matrix takes longer. But as an advantage, we do not have to use back-substitution. The rightmost column gives the solution directly.

Example 28–4: Solve the following system of equations:

$$\begin{aligned} y + 11z &= -3 \\ x + 3y + 10z &= 19 \\ x + y + 3z &= 10 \end{aligned}$$

Solution: The augmented matrix is:

$$\left[\begin{array}{ccc|c} 0 & 1 & 11 & -3 \\ 1 & 3 & 10 & 19 \\ 1 & 1 & 3 & 10 \end{array}\right]$$

Using row operations, we obtain:

$$R_1 \longleftrightarrow R_3 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 1 & 3 & 10 & 19 \\ 0 & 1 & 11 & -3 \end{array}\right]$$

$$R_2 \to R_2 - R_1 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 0 & 2 & 7 & 9 \\ 0 & 1 & 11 & -3 \end{array}\right]$$

$$R_2 \longleftrightarrow R_3 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 0 & 1 & 11 & -3 \\ 0 & 2 & 7 & 9 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 0 & 1 & 11 & -3 \\ 0 & 0 & -15 & 15 \end{array}\right]$$

$$R_3 \to -\frac{1}{15} R_3 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 0 & 1 & 11 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This is in REF but we continue:

$$R_2 \to R_2 - 11R_3 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$R_1 \to R_1 - 3R_3 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$R_1 \to R_1 - R_2 \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This augmented matrix is in RREF. The solution is:

$$\begin{aligned} x &= 5 \\ y &= 8 \\ z &= -1 \end{aligned}$$

Clearly, the system has unique solution.

Example 28–5: Solve the system of equations:

$$\begin{aligned} x_1 + x_2 - 2x_4 - 2x_5 &= 5 \\ -x_1 + 2x_2 + x_3 - x_4 - 5x_5 &= 7 \\ 3x_1 - 4x_3 + 6x_4 + x_5 &= 3 \end{aligned}$$

using Gauss-Jordan elimination.

Solution: The augmented matrix is:

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ -1 & 2 & 1 & -1 & -5 & 7 \\ 3 & 0 & -4 & 6 & 1 & 3 \end{array}\right]$$

Using row operations, we obtain:

$$R_2 \to R_2 + R_1 \qquad \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ 0 & 3 & 1 & -3 & -7 & 12 \\ 3 & 0 & -4 & 6 & 1 & 3 \end{array}\right]$$

$$R_3 \to R_3 - 3R_1 \qquad \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ 0 & 3 & 1 & -3 & -7 & 12 \\ 0 & -3 & -4 & 12 & 7 & -12 \end{array}\right]$$

$$R_2 \to \frac{1}{3}R_2 \qquad \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ 0 & 1 & \frac{1}{3} & -1 & -\frac{7}{3} & 4 \\ 0 & -3 & -4 & 12 & 7 & -12 \end{array}\right]$$

$$R_3 \to R_3 + 3R_2 \qquad \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ 0 & 1 & \frac{1}{3} & -1 & -\frac{7}{3} & 4 \\ 0 & 0 & -3 & 9 & 0 & 0 \end{array}\right]$$

$$R_3 \to -\frac{1}{3}R_3 \qquad \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ 0 & 1 & \frac{1}{3} & -1 & -\frac{7}{3} & 4 \\ 0 & 0 & 1 & -3 & 0 & 0 \end{array}\right]$$

This matrix is in REF but we continue the reduction:

$$R_2 \to R_2 - \tfrac{1}{3}R_3 \qquad \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ 0 & 1 & 0 & 0 & -\frac{7}{3} & 4 \\ 0 & 0 & 1 & -3 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - R_2 \qquad \left[\begin{array}{rrrrr|r} 1 & 0 & 0 & -2 & \frac{1}{3} & 1 \\ 0 & 1 & 0 & 0 & -\frac{7}{3} & 4 \\ 0 & 0 & 1 & -3 & 0 & 0 \end{array}\right]$$

This augmented matrix is in RREF.

$$\begin{aligned} x_1 - 2x_4 + \tfrac{1}{3}x_5 &= 1 \\ x_2 - \tfrac{7}{3}x_5 &= 4 \\ x_3 - 3x_4 &= 0 \end{aligned}$$

There are infinitely many solutions. We can choose x_4 and x_5 in an arbitrary way. The solution can be expressed as:

$$\begin{aligned} x_1 &= 1 + 2s - t \\ x_2 &= 4 + 7t \\ x_3 &= 3s \\ x_4 &= s \\ x_5 &= 3t \end{aligned}$$

where s and t are free parameters.

EXERCISES

Solve the following systems of equations:

28–1)
$$\begin{aligned} 2x+3y &= -6 \\ x-2y &= 11 \end{aligned}$$

28–2)
$$\begin{aligned} 2x+6y &= 24 \\ -x-3y &= -6 \end{aligned}$$

28–3)
$$\begin{aligned} 4x+5y &= -2 \\ -8x-10y &= 4 \end{aligned}$$

28–4)
$$\begin{aligned} 3x+5y &= 0 \\ x+5y &= -10 \end{aligned}$$

28–5)
$$\begin{aligned} 5x+2y &= 4 \\ 3x-y &= 20 \end{aligned}$$

28–6)
$$\begin{aligned} -x+3y &= 0 \\ 20x+10y &= 7 \end{aligned}$$

28–7)
$$\begin{aligned} x-y &= -4 \\ 10x-4y &= 2 \end{aligned}$$

28–8)
$$\begin{aligned} x+3y &= -15 \\ 3x-y &= 15 \end{aligned}$$

28–9)
$$\begin{aligned} 4x-8y &= 10 \\ 12x-24y &= 20 \end{aligned}$$

28–10)
$$\begin{aligned} x+5y &= 8 \\ 4x+20y &= 32 \end{aligned}$$

28–11)
$$\begin{aligned} 2x_1+6x_2 &= 13 \\ -x_1+4x_2 &= 11 \end{aligned}$$

28–12)
$$\begin{aligned} x_1-20x_2 &= 8 \\ 5x_1-x_2 &= 7 \end{aligned}$$

Solve the following systems of equations:

28–13)
$$\begin{aligned} x_1+2x_2-x_3 &= 6 \\ 2x_1+x_2+4x_3 &= 9 \\ -x_1-3x_2+5x_3 &= -5 \end{aligned}$$

28–14)
$$\begin{aligned} x_1+2x_2+2x_3 &= 11 \\ x_1+3x_2+13x_3 &= 10 \\ -x_1+2x_2+12x_3 &= 0 \end{aligned}$$

28–15)
$$\begin{aligned} x_1+4x_2+2x_3 &= -3 \\ x_1+x_2-x_3 &= 3 \\ -2x_1-4x_3 &= 6 \end{aligned}$$

28–16)
$$\begin{aligned} x_1+2x_2-x_3+x_4 &= 3 \\ 2x_1+x_2+x_3+x_4 &= 4 \\ x_1-x_2+2x_3 &= 1 \end{aligned}$$

28–17)
$$\begin{aligned} x_1+4x_3 &= 1 \\ 2x_1+x_2+3x_3 &= 5 \\ 3x_1+2x_2+2x_3 &= 9 \end{aligned}$$

28–18)
$$\begin{aligned} x_1+2x_2-5x_3 &= -1 \\ x_1+3x_2-7x_3 &= 0 \\ -x_1+x_2-2x_3 &= 3 \end{aligned}$$

28–19)
$$\begin{aligned} x_1-x_3 &= 3 \\ -x_1+2x_2-x_3+2x_4 &= -6 \\ 2x_1+3x_2+x_3 &= 9 \\ 4x_1+4x_3+10x_4 &= 15 \end{aligned}$$

28–20)
$$\begin{aligned} x_1+x_2+3x_3-x_4 &= 3 \\ -x_1+2x_2-3x_3+x_4 &= 0 \\ 5x_1+4x_2+10x_4 &= -1 \\ 7x_1+4x_2+6x_3+8x_4 &= 0 \end{aligned}$$

Solve the following homogeneous systems of equations:

28–21) $$\begin{aligned} 3x_1 + 2x_2 &= 0 \\ x_1 - 4x_2 &= 0 \end{aligned}$$

28–22) $$\begin{aligned} 2x_1 - 5x_2 &= 0 \\ -6x_1 + 15x_2 &= 0 \end{aligned}$$

28–23) $$\begin{aligned} x_1 - 3x_2 &= 0 \\ 4x_1 + 7x_2 &= 0 \\ 2x_1 + 8x_2 &= 0 \end{aligned}$$

28–24) $$\begin{aligned} x_1 - 4x_2 - 7x_3 &= 0 \\ -x_1 + 2x_2 + 5x_3 &= 0 \end{aligned}$$

28–25) $$\begin{aligned} x_1 - x_2 - x_3 &= 0 \\ 4x_1 + 2x_2 - 13x_3 &= 0 \\ 2x_1 + 4x_2 - 11x_3 &= 0 \end{aligned}$$

28–26) $$\begin{aligned} -x_1 + 2x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 - 6x_3 &= 0 \\ 2x_1 + x_2 + 7x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

28–27) $$\begin{aligned} 2x_1 + 3x_2 - 7x_3 - 7x_4 &= 0 \\ 3x_1 - 6x_2 + 21x_4 &= 0 \\ -x_1 - 5x_2 + 7x_3 + 14x_4 &= 0 \end{aligned}$$

28–28) $$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 2x_1 - 3x_2 - 9x_4 &= 0 \\ x_1 + 4x_2 + x_3 - x_4 &= 0 \end{aligned}$$

Solve the following systems of equations:

28–29) $$\begin{aligned} x + 2y + 3z &= 9 \\ 2y + z &= 4 \\ x + 2y + 4z &= 11 \end{aligned}$$

28–30) $$\begin{aligned} x + y - z &= 0 \\ 2x + 5y + z &= 9 \\ x + 8y + 4z &= 13 \end{aligned}$$

28–31) $$\begin{aligned} 2x - y + 5z &= -2 \\ 2y + 3z &= 16 \\ x + y - z &= 11 \end{aligned}$$

28–32) $$\begin{aligned} 3x + y - z &= 5 \\ x - 2y + 8z &= -3 \\ 10x + y + 5z &= 10 \end{aligned}$$

28–33) $$\begin{aligned} x + 2y &= 4 \\ x + 3y + z &= 3 \\ 4x + 7y - z &= 17 \end{aligned}$$

28–34) $$\begin{aligned} x - 5y + z &= 4 \\ 3x - 12y + 4z &= 9 \\ 3y + z &= 0 \end{aligned}$$

28–35) $$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 4 \\ 3x_1 - 2x_2 + 3x_3 - 2x_4 &= 15 \\ 2x_1 - 2x_2 + 3x_3 - x_4 &= 10 \\ 3x_1 - 3x_2 + 4x_3 - x_4 &= 16 \end{aligned}$$

ANSWERS

28–1) $x = 3, \quad y = -4.$

28–2) There's no solution.

28–3) There are infinitely many solutions. y is arbitrary parameter and $x = -\frac{1}{2} - \frac{5}{4}y.$

28–4) $x = 5, \quad y = -3.$

28–5) $x = 4, \quad y = -8.$

28–6) $x = 0.3, \quad y = 0.1.$

28–7) $x = 3, \quad y = 7.$

28–8) $x = 3, \quad y = -6.$

28–9) There's no solution.

28–10) There are infinitely many solutions. y is arbitrary parameter and $x = 8 - 5y.$

28–11) $x_1 = -1, \quad x_2 = 2.5.$

28–12) $x_1 = \frac{4}{3}, \quad x_2 = -\frac{1}{3}.$

28–13) $x_1 = 1, \quad x_2 = 3, \quad x_3 = 1.$

28–14) $x_1 = 3, \quad x_2 = 4.5, \quad x_3 = -0.5.$

28–15) $x_1 = 1, \quad x_2 = 0, \quad x_3 = -2.$

28–16) There are infinitely many solutions given by:
$x_1 = \frac{5}{3} - s - r, \quad x_2 = \frac{2}{3} - s + r, \quad x_3 = r, \quad x_4 = 3s.$

28–17) There are infinitely many solutions. z is arbitrary parameter and $x = 1 - 4r, \quad y = 3 + 5r.$

28–18) $x_1 = -2, \quad x_2 = 3, \quad x_3 = 1.$

28–19) $x_1 = 4, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = -0.5.$

28–20) There's no solution.

28–21) Trivial solution.

28–22) $x_1 = 5r, \quad x_2 = 2r.$

28–23) Trivial solution.

28–24) $x_1 = 3r, \quad x_2 = -r, \quad x_3 = r.$

28–25) $x_1 = 5r, \quad x_2 = 3r, \quad x_3 = 2r.$

28–26) Trivial solution.

28–27) $x_1 = 2r - s, \quad x_2 = r + 3s, \quad x_3 = r, \quad x_4 = s.$

28–28) $x_1 = 3r, \quad x_2 = -r, \quad x_3 = 2r, \quad x_4 = r.$

28–29) $x = 1, \quad y = 1, \quad z = 2.$ Unique Solution.

28–30) $x = 5, \quad y = -1, \quad z = 4.$ Unique Solution.

28–31) $x = 3, \quad y = 8, \quad z = 0.$ Unique Solution.

28–32) No Solution.

28–33) $x = 6 + 2z, \quad y = -z - 1, \quad z$ is a free parameter.
Infinitely Many Solutions.

28–34) No Solution.

28–35) $x_1 = 7, \quad x_2 = 1, \quad x_3 = 0, \quad x_4 = 2.$
Unique Solution.

Chapter 29

Inverse Matrices

Let A be an $n \times n$ matrix. If there exists another $n \times n$ matrix B such that

$$AB = BA = I$$

then B is called the **inverse** of A. (Similarly, A is the inverse of B.) For example,

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -\frac{1}{2} \\ -3 & 1 \end{bmatrix}$$

are inverses of each other.

If a square matrix A has an inverse, we say A is **invertible**.

If A and B are invertible and same size, then

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^{-1} = A$
- $(A^n)^{-1} = (A^{-1})^n = A^{-n}$
- $(A^T)^{-1} = (A^{-1})^T$

Example 29–1: Find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if it exists.

Solution: Assume the inverse exists and it is $A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$.

Using definition, we obtain:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} aw + by &= 1 \\ cw + dy &= 0 \end{aligned}$$

$$\begin{aligned} ax + bz &= 0 \\ cx + dz &= 1 \end{aligned}$$

We can solve for w, x, y, z using elimination.

The solution is: $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Therefore, the inverse of A exists if $ad - bc \neq 0$.

Matrix Inverse by Row Reduction:
Let A be a square matrix. To find A^{-1}, write the identity matrix I next to A:

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

Then use row operations to obtain:

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

(This is equivalent to finding RREF.)

If there is a row of zeros on the bottom (for the left part), that means A can not be reduced to I. In other words A is not invertible.

Example 29–2: Find the inverse of

$$A = \begin{bmatrix} 1 & 4 & 21 \\ 0 & -1 & -5 \\ 1 & 2 & 10 \end{bmatrix}$$

if it exists.

Solution: Let's start with:

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 21 & 1 & 0 & 0 \\ 0 & -1 & -5 & 0 & 1 & 0 \\ 1 & 2 & 10 & 0 & 0 & 1 \end{array}\right]$$

and use row reduction:

$$R_3 \to R_3 - R_1 \qquad \left[\begin{array}{ccc|ccc} 1 & 4 & 21 & 1 & 0 & 0 \\ 0 & -1 & -5 & 0 & 1 & 0 \\ 0 & -2 & -11 & -1 & 0 & 1 \end{array}\right]$$

$$R_2 \to -R_2 \qquad \left[\begin{array}{ccc|ccc} 1 & 4 & 21 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & -1 & 0 \\ 0 & -2 & -11 & -1 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 + 2R_2 \qquad \left[\begin{array}{ccc|ccc} 1 & 4 & 21 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & -2 & 1 \end{array}\right]$$

$$R_3 \to -R_3 \qquad \left[\begin{array}{ccc|ccc} 1 & 4 & 21 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right]$$

$$R_2 \to R_2 - 5R_3 \qquad \left[\begin{array}{ccc|ccc} 1 & 4 & 21 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5 & -11 & 5 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right]$$

$$R_1 \to R_1 - 21R_3 \qquad \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & -20 & -42 & 21 \\ 0 & 1 & 0 & -5 & -11 & 5 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right]$$

$$R_1 \to R_1 - 4R_2 \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -5 & -11 & 5 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right]$$

Therefore $A^{-1} = \begin{bmatrix} 0 & 2 & 1 \\ -5 & -11 & 5 \\ 1 & 2 & -1 \end{bmatrix}$.

Example 29–3: Find the inverse of $B = \begin{bmatrix} 1 & 2 & -2 & -1 \\ 1 & 3 & 5 & 7 \\ 0 & 0 & 2 & 2 \\ 2 & 4 & 1 & 4 \end{bmatrix}$.

Solution: Start with: $\left[\begin{array}{cccc|cccc} 1 & 2 & -2 & -1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 5 & 7 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 1 & 0 \\ 2 & 4 & 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$

$$\begin{array}{l} R_2 \to R_2 - R_1 \\ R_4 \to R_4 - 2R_1 \end{array} \qquad \left[\begin{array}{cccc|cccc} 1 & 2 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 8 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 6 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_3 \to \frac{1}{2} R_3 \\ R_4 \to R_4 - 5R_3 \end{array} \qquad \left[\begin{array}{cccc|cccc} 1 & 2 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 8 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & -\frac{5}{2} & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 + R_4 \\ R_2 \to R_2 - 8R_4 \\ R_3 \to R_3 - R_4 \end{array} \qquad \left[\begin{array}{cccc|cccc} 1 & 2 & -2 & 0 & -1 & 0 & -\frac{5}{2} & 1 \\ 0 & 1 & 7 & 0 & 15 & 1 & 20 & -8 \\ 0 & 0 & 1 & 0 & 2 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 0 & -\frac{5}{2} & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 - 7R_3 \\ R_1 \to R_1 - 2R_2 \end{array} \qquad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & \frac{11}{2} & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 0 & -\frac{5}{2} & 1 \end{array}\right]$$

$\Rightarrow$ B^{-1} is the right part of the matrix.

Example 29–4: Find the inverse of

$$C = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 5 & -6 \\ 7 & 8 & -12 \end{bmatrix}$$

if it exists.

Solution: Let's start with:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 3 & 5 & -6 & 0 & 1 & 0 \\ 7 & 8 & -12 & 0 & 0 & 1 \end{array}\right]$$

and use row reduction:

$$R_2 \to R_2 - 3R_1 \qquad \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 11 & -6 & -3 & 1 & 0 \\ 7 & 8 & -12 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 - 7R_1 \qquad \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 11 & -6 & -3 & 1 & 0 \\ 0 & 22 & -12 & -7 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2 \qquad \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 11 & -6 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$$

We have a row of zeros.

$\Rightarrow$ It is impossible to row reduce C to I.

$\Rightarrow$ C is not invertible.

Solution of Systems of Equations by Matrix Inverses:
Consider the system of n linear equations in m unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ \vdots \quad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n \end{aligned}$$

We can represent this system as a matrix equation:

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

For the special case $n = m$, the matrix A is square. In this case, there is a unique solution if and only if A is invertible, where

$$\vec{x} = A^{-1}\vec{b}$$

If A is not invertible, there may be no solution or there may be infinitely many solutions, depending on $\vec{b}$.
The homogeneous system $A\vec{x} = \vec{0}$ always has the trivial (zero) solution.

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

If A is invertible, there is no other, nontrivial solution.

Example 29–5: Solve the following systems of equations:

$$\begin{aligned} x_1 + 4x_3 &= 6 \\ -2x_2 - x_3 &= 5 \\ 3x_1 + 5x_2 + 2x_3 &= -7 \end{aligned} \qquad \begin{aligned} y_1 + 4y_3 &= 50 \\ -2y_2 - y_3 &= 75 \\ 3y_1 + 5y_2 + 2y_3 &= 100 \end{aligned}$$

Solution: We can solve multiple systems having the same coefficient matrix by using a single augmented matrix. The given systems can be written as:

$A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{c}$, where

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -1 \\ 3 & 5 & 2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 6 \\ 5 \\ -7 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 50 \\ 75 \\ 100 \end{bmatrix}.$$

Using row reduction on $\left[\, A \,\middle|\, I \,\right]$ we find:

$$A^{-1} = \frac{1}{25}\begin{bmatrix} 1 & 20 & 8 \\ -3 & -10 & 1 \\ 6 & -5 & -2 \end{bmatrix}.$$

Now we can solve both systems easily as:

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \text{ and}$$

$$\vec{y} = A^{-1}\vec{c} = \begin{bmatrix} 94 \\ -32 \\ -11 \end{bmatrix}$$

EXERCISES

Find the inverse of each matrix, if it exists:

29–1) $\begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix}$

29–2) $\begin{bmatrix} 13 & 5 \\ 5 & 1 \end{bmatrix}$

29–3) $\begin{bmatrix} 4 & 3 \\ 12 & 9 \end{bmatrix}$

29–4) $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

29–5) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & 5 & 2 \end{bmatrix}$

29–6) $\begin{bmatrix} 4 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

29–7) $\begin{bmatrix} 7 & 2 & 3 \\ -4 & 1 & -5 \\ -1 & 4 & -7 \end{bmatrix}$

29–8) $\begin{bmatrix} 11 & 6 & -5 \\ -8 & 2 & 12 \\ 1 & -9 & -13 \end{bmatrix}$

Find the inverse of each matrix, if it exists:

29–9) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

29–10) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

29–11) $\begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ -4 & -2 & 1 \end{bmatrix}$

29–12) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & 11 & 18 \\ 2 & 6 & 11 \end{bmatrix}$

29–13) $\begin{bmatrix} 2 & 4 & 1 \\ 6 & 6 & -20 \\ 1 & 5 & 12 \end{bmatrix}$

29–14) $\begin{bmatrix} 1 & 4 & 6 \\ -1 & 2 & 8 \\ 5 & 3 & 1 \end{bmatrix}$

29–15) $\begin{bmatrix} 5 & 2 & 0 \\ 1 & -2 & 1 \\ 4 & 3 & -1 \end{bmatrix}$

29–16) $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ 1 & 0 & -5 \end{bmatrix}$

Find the inverse of each matrix, if it exists:

29–17) $\begin{bmatrix} 4 & 0 & 0 \\ -2 & 3 & 0 \\ 1 & -9 & 1/2 \end{bmatrix}$

29–18) $\begin{bmatrix} 0.5 & -1 & 5 \\ 1 & 2 & 4 \\ 1 & 6 & 0 \end{bmatrix}$

29–19) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 1 & 3 & -2 \\ 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 6 \end{bmatrix}$

29–20) $\begin{bmatrix} 2 & -2 & 4 & 1 \\ 2 & -1 & 0 & 4 \\ 2 & -3 & 6 & 1 \\ 0 & 1 & 0 & -2 \end{bmatrix}$

29–21) $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & 1 & 3 & -1 \\ 1 & 1 & 2 & 1 \\ -2 & 0 & 1 & 5 \end{bmatrix}$

29–22) $\begin{bmatrix} 0 & 2 & -2 & 0 \\ 0 & 1 & 2 & 4 \\ 2 & 1 & 0 & 4 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

Solve the following systems of equations using inverse of the coefficient matrix:

29–23) $\begin{aligned} 3x_1 + 4x_2 - 2x_3 &= 11 \\ -x_1 + 2x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$

29–24) $\begin{aligned} 3x_1 + 4x_2 - 2x_3 &= 16 \\ -x_1 + 2x_2 + 2x_3 &= 18 \\ x_1 + 2x_2 - x_3 &= 7 \end{aligned}$

29–25) $\begin{aligned} 5x_1 + 8x_2 + 5x_3 &= 9 \\ x_1 + 2x_2 + x_3 &= 3 \\ x_2 + x_3 &= 4 \end{aligned}$

29–26) $\begin{aligned} 5x_1 + 8x_2 + 5x_3 &= 2 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_2 + x_3 &= 1 \end{aligned}$

29–27) $\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 5x_2 + x_3 &= 15 \\ -x_1 + x_3 &= -2 \end{aligned}$

29–28) $\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ -x_1 - x_3 &= 0 \\ x_1 + 5x_2 + 2x_3 &= -2 \end{aligned}$

29–29) $\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 + x_2 + 3x_3 - 2x_4 &= -1 \\ x_1 - x_2 + 2x_3 - 3x_4 &= -11 \\ x_2 - 2x_3 + 6x_4 &= 25 \end{aligned}$

29–30) $\begin{aligned} 2x_1 - 2x_2 + 4x_3 + x_4 &= 5 \\ 2x_1 - x_2 + 4x_4 &= 9 \\ 2x_1 - 3x_2 + 5x_3 + x_4 &= 4 \\ x_2 - 2x_4 &= -1 \end{aligned}$

ANSWERS

29–1) $\frac{1}{8}\begin{bmatrix} -2 & 5 \\ 2 & -1 \end{bmatrix}$

29–2) $\frac{1}{12}\begin{bmatrix} -1 & 5 \\ 5 & -13 \end{bmatrix}$

29–3) Inverse does not exist.

29–4) $\frac{1}{3}\begin{bmatrix} -5 & 2 & -2 \\ 4 & -1 & 1 \\ -5 & 2 & 1 \end{bmatrix}$

29–5) $\begin{bmatrix} 2.5 & 0.5 & -1 \\ 0.5 & 0.5 & 0 \\ -2.5 & -1.5 & 1 \end{bmatrix}$

29–6) $\frac{1}{10}\begin{bmatrix} -2 & 0 & 6 \\ -2 & 5 & 1 \\ 6 & 0 & -8 \end{bmatrix}$

29–7) Inverse does not exist.

29–8) Inverse does not exist.

29–9) $\frac{1}{2}\begin{bmatrix} 6 & -2 & -4 \\ 1 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix}$

29–10) $\frac{1}{2}\begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & -1 \\ -1 & 2 & -1 \end{bmatrix}$

29–11) $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix}$

29–12) $\frac{1}{6}\begin{bmatrix} 13 & -9 & 10 \\ 3 & 3 & -6 \\ -4 & 0 & 2 \end{bmatrix}$

29–13) Inverse does not exist.

29–14) $\frac{1}{64}\begin{bmatrix} -22 & 14 & 20 \\ 41 & -29 & -14 \\ -13 & 17 & 6 \end{bmatrix}$

29–15) $\frac{1}{5}\begin{bmatrix} -1 & 2 & 2 \\ 5 & -5 & -5 \\ 11 & -7 & -12 \end{bmatrix}$

29–16) Inverse does not exist.

29–17) $\frac{1}{12}\begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ 30 & 72 & 24 \end{bmatrix}$

29–18) $\frac{1}{4}\begin{bmatrix} -24 & 30 & -14 \\ 4 & -5 & 3 \\ 4 & -4 & 2 \end{bmatrix}$

29–19) $\frac{1}{7}\begin{bmatrix} 15 & -12 & 28 & 10 \\ -4 & 6 & -14 & -5 \\ -17 & 15 & -28 & -9 \\ -5 & 4 & -7 & -1 \end{bmatrix}$

29–20) $\frac{1}{2}\begin{bmatrix} 6 & -1 & -4 & -1 \\ -12 & 4 & 8 & 6 \\ -7 & 2 & 5 & 3 \\ -6 & 2 & 4 & 2 \end{bmatrix}$

29–21) Inverse does not exist.

29–22) $\frac{1}{18}\begin{bmatrix} -7 & -7 & 5 & 8 \\ 2 & 2 & -4 & 8 \\ -7 & 2 & -4 & 8 \\ 3 & 3 & 3 & -6 \end{bmatrix}$

29–23) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

29–24) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix}$

29–25) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$

29–26) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

29–27) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

29–28) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$

29–29) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}$

29–30) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Chapter 30

Determinants

Determinants by Cofactors:
The determinant of a 1×1 matrix is itself. For any other square matrix, we define the determinant recursively.
We will use the notation $\left|A\right|$ to denote the determinant of the square matrix A.

Cofactor: Let A be an $n \times n$ matrix. Let's delete the i^{th} row and j^{th} column, and calculate the determinant of the remaining $(n-1) \times (n-1)$ submatrix. Call it M_{ij}. The cofactor of the entry a_{ij} is:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For example, for the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

cofactor of a is d and the cofactor of b is $-c$.
For the matrix

$$\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$$

cofactor of a is $\begin{vmatrix} q & r \\ y & z \end{vmatrix}$, cofactor of b is $-\begin{vmatrix} p & r \\ x & z \end{vmatrix}$ and cofactor of q is $\begin{vmatrix} a & c \\ x & z \end{vmatrix}$.

The determinant of any other square matrix is defined as:

$$\left|A\right| = \det(A) = a_{i1}C_{i1} + \cdots + a_{in}C_{in}, \quad \text{or}$$
$$\left|A\right| = \det(A) = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}$$

In other words:

- Choose a row (or column).
- Find the cofactor of each entry on that row (or column).
- Multiply the entries by the cofactors and add.

This is called the cofactor expansion. For example, the determinant of a 2×2 matrix is:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(Do you remember this expression from previous chapters?)

If we use the first row to find the determinant of a 3×3 matrix, we obtain:

$$\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = a\begin{vmatrix} q & r \\ y & z \end{vmatrix} - b\begin{vmatrix} p & r \\ x & z \end{vmatrix} + c\begin{vmatrix} p & q \\ x & y \end{vmatrix}$$
$$= aqz - ayr - bpz + bxr + cpy - cqx$$

If we choose the second column, we obtain:

$$\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -b\begin{vmatrix} p & r \\ x & z \end{vmatrix} + q\begin{vmatrix} a & c \\ x & z \end{vmatrix} - y\begin{vmatrix} a & c \\ p & r \end{vmatrix}$$
$$= -bpz + bxr + aqz - cqx - ayr + cpy$$

which is identical. It does not matter which one we choose. Note that in each term, there's one and only one entry from each row and each column. Therefore the determinant of an $n \times n$ matrix requires $n!$ terms.

Example 30–1: Find the determinant of $A = \begin{bmatrix} 4 & 9 & 5 \\ 1 & 2 & 0 \\ -3 & 6 & 0 \end{bmatrix}$.

Solution: Using the third column, we obtain:

$$|A| = 5\begin{vmatrix} 1 & 2 \\ -3 & 6 \end{vmatrix}.$$
$$= 5\big(6 - (-6)\big)$$
$$= 60$$

Example 30–2: Find determinant of $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 0 & 0 & 4 \\ 8 & -1 & 3 & 0 \\ -5 & 1 & 2 & -7 \end{bmatrix}$.

Solution: Using the second row, we obtain:

$$|A| = -2\begin{vmatrix} 2 & -1 & 3 \\ -1 & 3 & 0 \\ 1 & 2 & -7 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 & -1 \\ 8 & -1 & 3 \\ -5 & 1 & 2 \end{vmatrix}.$$
$$= (-2)\cdot(-50) + 4\cdot(-70)$$
$$= -180$$

Example 30–3: Find the determinant of $A = \begin{bmatrix} 5 & 48 & 7 & 12 \\ 0 & 3 & 10 & -3 \\ 0 & 0 & 2 & 99 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution: Using the first column, we obtain:

$$|A| = 5\begin{vmatrix} 3 & 10 & -3 \\ 0 & 2 & 99 \\ 0 & 0 & 1 \end{vmatrix}.$$
$$= 5\cdot 3\begin{vmatrix} 2 & 99 \\ 0 & 1 \end{vmatrix}$$
$$= 5\cdot 3\cdot 2\cdot 1$$
$$= 30$$

Only the diagonal entries matter.

Inverses by Adjoint Matrices: Let A be an $n \times n$ matrix and C_{ij} be the cofactor of a_{ij}. If we replace each entry by its cofactor and then transpose the matrix, we obtain the adjoint. In other words, the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$$

is called the **adjoint** of A.
Do not forget that we have a sign of $(-1)^{i+j}$ in front of each entry. These signs are arranged as follows:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

For example, the adjoint of $A = \begin{bmatrix} 8 & 4 & 0 \\ 1 & 5 & 7 \\ 2 & 3 & 6 \end{bmatrix}$ is:

$$\begin{bmatrix} 9 & 8 & -7 \\ -24 & 48 & -16 \\ 28 & -56 & 36 \end{bmatrix}^T = \begin{bmatrix} 9 & -24 & 28 \\ 8 & 48 & -56 \\ -7 & -16 & 36 \end{bmatrix}$$

Theorem: If A is an invertible matrix, then $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$

Example 30–4: Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ using the above formula.

Solution:

$$\begin{aligned} A^{-1} &= \frac{\operatorname{adj}(A)}{\det(A)} \\ &= \frac{\begin{bmatrix} \begin{vmatrix} 5 & 1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} \end{bmatrix}^T}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 0 & 1 & 4 \end{vmatrix}} \\ &= \frac{\begin{bmatrix} 19 & -8 & 2 \\ -5 & 4 & -1 \\ -13 & 5 & 1 \end{bmatrix}^T}{9} \\ &= \frac{1}{9} \begin{bmatrix} 19 & -5 & -13 \\ -8 & 4 & 5 \\ 2 & -1 & 1 \end{bmatrix} \end{aligned}$$

Example 30–5: Find the inverse of the matrix $B = \begin{bmatrix} 7 & 5 & 8 \\ 2 & 1 & 0 \\ 6 & 4 & 3 \end{bmatrix}$

Solution: Let's find the cofactor of each entry:

Cofactor of 7: $C_{11} = \begin{vmatrix} 1 & 0 \\ 4 & 3 \end{vmatrix} = 3$

Cofactor of 5: $C_{12} = -\begin{vmatrix} 2 & 0 \\ 6 & 3 \end{vmatrix} = -6$

Cofactor of 8: $C_{13} = \begin{vmatrix} 2 & 1 \\ 6 & 4 \end{vmatrix} = 2$

Cofactor of 2: $C_{21} = -\begin{vmatrix} 5 & 8 \\ 4 & 3 \end{vmatrix} = 17$

Cofactor of 1: $C_{22} = \begin{vmatrix} 7 & 8 \\ 6 & 3 \end{vmatrix} = -27$

Cofactor of 0: $C_{23} = -\begin{vmatrix} 7 & 5 \\ 6 & 4 \end{vmatrix} = 2$

Cofactor of 6: $C_{31} = \begin{vmatrix} 5 & 8 \\ 1 & 0 \end{vmatrix} = -8$

Cofactor of 4: $C_{32} = -\begin{vmatrix} 7 & 8 \\ 2 & 0 \end{vmatrix} = 16$

Cofactor of 3: $C_{33} = \begin{vmatrix} 7 & 5 \\ 2 & 1 \end{vmatrix} = -3$

The determinant of B is:

$$\begin{aligned} |B| &= 7\begin{vmatrix} 1 & 0 \\ 4 & 3 \end{vmatrix} - 5\begin{vmatrix} 2 & 0 \\ 6 & 3 \end{vmatrix} + 8\begin{vmatrix} 2 & 1 \\ 6 & 4 \end{vmatrix} \\ &= 21 - 30 + 16 = 7 \end{aligned}$$

The adjoint of B is:

$$\begin{aligned} \operatorname{adj} B &= \begin{bmatrix} 3 & -6 & 2 \\ 17 & -27 & 2 \\ -8 & 16 & -3 \end{bmatrix}^T \\ &= \begin{bmatrix} 3 & 17 & -8 \\ -6 & -27 & 16 \\ 2 & 2 & -3 \end{bmatrix} \end{aligned}$$

Therefore the inverse of B is:

$$B^{-1} = \frac{\operatorname{adj}(B)}{\det(B)} = \frac{1}{7}\begin{bmatrix} 3 & 17 & -8 \\ -6 & -27 & 16 \\ 2 & 2 & -3 \end{bmatrix}$$

Properties of the Determinant Function:
For $n \times n$ matrices:

- If A has a row (or column) of zeros, then $\det(A) = 0$.
- If A is an upper triangular, lower triangular or a diagonal matrix, then $\det(A)$ is the product of the entries on the main diagonal.
- The determinant of any identity matrix is 1.
- $\det(A) = \det(A^T)$
- A is invertible if and only if $\det(A) \neq 0$.
- $\det(AB) = \det(A)\det(B) \quad \Rightarrow \quad \det(A^{-1}) = \dfrac{1}{\det(A)}$.

EXERCISES

Find the determinants of the following matrices: (if defined)

30–1) $\begin{bmatrix} 3 & 7 \\ -2 & 6 \end{bmatrix}$

30–2) $\begin{bmatrix} 10 & 29 \\ 0 & 17 \end{bmatrix}$

30–3) $\begin{bmatrix} 5 & 9 & 7 \\ 12 & 0 & -3 \end{bmatrix}$

30–4) $\begin{bmatrix} 5 & 1 & 0 \\ 0 & 1 & 4 \\ 3 & 2 & -1 \end{bmatrix}$

30–5) $\begin{bmatrix} 8 & 2 & -1 \\ 1 & 0 & 0 \\ 9 & 6 & -5 \end{bmatrix}$

30–6) $\begin{bmatrix} -5 & 6 & 3 \\ 0 & 7 & 8 \\ 1 & 2 & 3 \end{bmatrix}$

30–7) $\begin{bmatrix} 3 & 1 & -2 & 0 \\ 5 & 1 & 3 & 0 \\ 4 & 2 & 1 & 6 \\ 8 & 2 & 3 & 0 \end{bmatrix}$

30–8) $\begin{bmatrix} 7 & 12 & 9 & 20 \\ 0 & 1 & 6 & 8 \\ 0 & 0 & 2 & 16 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Find the determinants of the following matrices: (if defined)

30–9) $\begin{bmatrix} 5 & x \\ 3 & 6 \end{bmatrix}$

30–10) $\begin{bmatrix} 4 & 13 & 2 \\ 0 & 1 & 0 \\ 8 & -7 & 9 \end{bmatrix}$

30–11) $\begin{bmatrix} a & b & c \\ 6 & 3 & -1 \\ 4 & 2 & 0 \end{bmatrix}$

30–12) $\begin{bmatrix} 7 & 5 & 3 \\ 14 & 10 & 6 \\ -4 & -2 & 1 \end{bmatrix}$

30–13) $\begin{bmatrix} 3 & 8 & 0 \\ x & y & z \\ 2 & 5 & 0 \end{bmatrix}$

30–14) $\begin{bmatrix} 2 & 4 & 0 & 3 \\ 0 & 7 & -8 & 12 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

30–15) $\begin{bmatrix} 7 & 11 \\ 56 & 93 \end{bmatrix}$

30–16) $\begin{bmatrix} 1 & 7 & 8 \\ 2 & 16 & 25 \\ 6 & 48 & 79 \end{bmatrix}$

Find the inverse of the following matrices using the adjoint formula:
(If possible)

30–17) $\begin{bmatrix} 10 & 8 \\ 7 & 6 \end{bmatrix}$

30–18) $\begin{bmatrix} 6 & 11 \\ 4 & 9 \end{bmatrix}$

30–19) $\begin{bmatrix} 10 & 7 \\ 30 & 21 \end{bmatrix}$

30–20) $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix}$

30–21) $\begin{bmatrix} 5 & 0 & 2 \\ 1 & -8 & 9 \\ 4 & -2 & 3 \end{bmatrix}$

30–22) $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 3 & 4 \\ 2 & 0 & 3 \end{bmatrix}$

30–23) $\begin{bmatrix} 0 & 1 & 0 \\ 5 & 0 & -4 \\ 0 & 2 & 0 \end{bmatrix}$

30–24) $\begin{bmatrix} 2 & 0 & 1 & 0 \\ 3 & 5 & 0 & -1 \\ 1 & 0 & 2 & 0 \\ 0 & 4 & -2 & 1 \end{bmatrix}$

ANSWERS

30–1) 32

30–2) 170

30–3) Determinant is defined only for square matrices.

30–4) -33

30–5) 4

30–6) 2

30–7) 24

30–8) 14

30–9) $30 - 3x$

30–10) 20

30–11) $2a - 4b$

30–12) 0

30–13) z

30–14) 28

30–15) 35

30–16) 8

30–17) $\frac{1}{4}\begin{bmatrix} 6 & -8 \\ -7 & 10 \end{bmatrix}$

30–18) $\frac{1}{10}\begin{bmatrix} 9 & -11 \\ -4 & 6 \end{bmatrix}$

30–19) Inverse does not exist.

30–20) $\frac{1}{24}\begin{bmatrix} -12 & 12 & 6 \\ 18 & -6 & -11 \\ 0 & 0 & 4 \end{bmatrix}$

30–21) $\frac{1}{30}\begin{bmatrix} -6 & -4 & 16 \\ 33 & 7 & -43 \\ 30 & 10 & -40 \end{bmatrix}$

30–22) $\frac{1}{15}\begin{bmatrix} -9 & 0 & 3 \\ -8 & 5 & -4 \\ 6 & 0 & 3 \end{bmatrix}$

30–23) Inverse does not exist.

30–24) $\frac{1}{27}\begin{bmatrix} 18 & 0 & -9 & 0 \\ -8 & 3 & 7 & 3 \\ -9 & 0 & 18 & 0 \\ 14 & -12 & 8 & 15 \end{bmatrix}$