10.4: Substitution: $f(u(x))$

Chain rule: $u=u(x)$

$$\frac{d}{dx}F(u(x)) = \boxed{\frac{dF(u)}{du}} \cdot \frac{du(x)}{dx}$$

integrating both sides: for $\boxed{f=F'}$

$$\underbrace{\int \frac{d}{dx}F(u(x))dx}_{\boxed{F(u(x))+C}} = \int f(u(x)).\,u'(x)\,dx$$

$$\Rightarrow \boxed{\int f(u(x)).\,u'(x)\,dx = \int f(u)\,du}$$

$$\int f(u)\,du = F(u)+C = F(u(x))+C$$

where F is an antiderivative of f

eg. $\displaystyle\int \frac{2x\,dx}{x^2+1} = \int \frac{du}{u} = \ln|u| + C = \boxed{\ln(x^2+1)+C}$

substition: $u=x^2+1$ $\Rightarrow$ simplifies the integral

$\boxed{\frac{du}{dx}=2x} \Rightarrow \boxed{du=2x\,dx}$

Ex 10.7: Evaluate the integral $\displaystyle\int (\underbrace{x^4+1}_{u})^2 \cdot 4x^3\,dx = ?$

Soln.: $\boxed{\begin{aligned} u &= x^4+1 \\ du &= 4x^3\,dx \end{aligned}}$

$$\int (x^4+1)^2(4x^3\,dx) = \int u^2\,du = \frac{u^3}{3}+C = \boxed{\frac{(x^4+1)^3}{3}+C}$$

Ex.10-8: $\int e^{3x^2} \cdot x\,dx = \int e^{u}\frac{1}{6}\,du = \frac{1}{6}\int e^{u}\,du = \boxed{\frac{1}{6}e^{u}+C}$

$u = 3x^2$

$du = 6x\,dx$

$\frac{1}{6}du = x\,dx$

$= \boxed{\frac{1}{6}e^{3x^2} + C}$

Check: $\frac{d}{dx}\left(\frac{1}{6}e^{3x^2}+C\right) = \frac{1}{\cancel{6}}e^{3x^2}\cdot \cancel{6}x + 0 = \boxed{e^{3x^2}\cdot x}$ *

Ex.10-9: $\int (x^3+6x^2)^7 \cdot \boxed{(x^2+4x)\,dx} =$

$\boxed{u = x^3+6x^2}$

$du = (3x^2+12x)\,dx$

$= 3\boxed{(x^2+4x)\,dx}$

$\boxed{\frac{1}{3}du = (x^2+4x)\,dx}$

$= \int u^7 \cdot \frac{1}{3}\,du$

$= \frac{1}{3}\int u^7\,du = \frac{1}{3}\frac{u^8}{8} + C$

$= \boxed{\frac{1}{24}(x^3+6x^2)^8 + C}$

10.5: Substitution in Definite Integrals:

F.T.C: $\int_a^b f(t)\,dt = F(t)\Big|_a^b = F(b)-F(a)$

$\boxed{f' = F}$

$$\int_a^b f(u(x))\cdot u'(x) = \int_{u(a)}^{u(b)} f(u)\,du \qquad u=u(x)$$

* In definite integrals when you use the substitution: $u=u(x)$, DO NOT FORGET TO CHANGE THE INTEGRATION BOUNDS:

$$\left.\begin{matrix} x=a \\ x=b \end{matrix}\right\} \Longrightarrow \left\{\begin{matrix} u=u(a) \\ u=u(b) \end{matrix}\right.$$

integration limits

Ex.10-10: $\displaystyle\int_{x=1}^{x=2} \frac{x+2}{x^2+4x+1}\,dx = \int_{u=6}^{u=13} \frac{\frac{1}{2}du}{u} = \frac{1}{2}\int_{u=6}^{u=13} \frac{du}{u} = \frac{1}{2}\ln|u| \Big|_{u=6}^{u=13}$

$\boxed{u = x^2+4x+1} \Rightarrow \begin{cases} x=2 \Rightarrow u \neq 2^2+4(2)+1 = 13 \\ x=1 \Rightarrow u = 1^2+4(1)+1 = 6 \end{cases}$

$du = (2x+4)\,dx$

$= 2(x+2)\,dx$

$\boxed{\frac{1}{2}du = (x+2)\,dx}$

$= \frac{1}{2}(\ln 13 - \ln 6) = \boxed{\frac{1}{2}\ln\left(\frac{13}{6}\right)}$ ✓

*Note: if we forget to change the integration limits (bounds) from $\left.\begin{matrix} x=1 \\ x=2 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} u=6 \\ u=13 \end{matrix}\right\}$ we would get:

$\cancel{\displaystyle\int_1^2 \frac{1}{2}\frac{du}{u} = \frac{1}{2}\ln u \Big|_1^2 = \frac{1}{2}(\ln(2) - \ln 1) = \frac{1}{2}\ln 2 = \boxed{\ln\sqrt{2}}}$

$\ln 1 = 0$

Ex 10-12: $\displaystyle\int_{\boxed{x=-4}}^{x=4} \frac{x}{\sqrt{5-x}}\,dx$

$\boxed{u=5-x} \Rightarrow \begin{cases} x=-4 \Rightarrow u=5-(-4)=9 \\ x=4 \Rightarrow u=5-4=1 \end{cases}$

$x=5-u$

$\boxed{du=-dx} \Rightarrow \boxed{-du=dx}$

$$= \int_{u=9}^{u=1} \frac{5-u}{\sqrt{u}}(-du) = +\int_1^9 \left(5u^{-1/2} - u^{1/2}\right)du = \left[5\frac{u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2}\right]_1^9$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$= \left[10\sqrt{u} - \frac{2}{3}(\sqrt{u})^3\right]_1^9$$

$$= \left(10\sqrt{9} - \frac{2}{3}(\sqrt{9})^3\right) - \left(10\sqrt{1} - \frac{2}{3}(\sqrt{1})^3\right)$$

$$= (30-18) - \left(10 - \frac{2}{3}\right) = 2 + \frac{2}{3} = \boxed{\frac{8}{3}}$$

Ex 10-13: $\displaystyle\int_0^2 8\underbrace{e^{\frac{x^4}{5}}}_{e^u} \cdot \underbrace{x^3dx}_{\frac{5}{4}du} = ?$

$\boxed{u=\frac{x^4}{5}} \Rightarrow \begin{cases} x=2 \Rightarrow u=\frac{16}{5} \\ x=0 \Rightarrow u=0 \end{cases}$

$du = \frac{4x^3}{5}dx$

$\boxed{\frac{5}{4}du = x^3dx}$

$$= \int 8e^u \cdot \frac{5}{4}\,du$$

$$= 10\int_{u=0}^{u=\frac{16}{5}} e^u\,du = 10(e^u)\Big|_0^{\frac{16}{5}} = 10\left(e^{\frac{16}{5}} - \underset{1}{e^0}\right) = \boxed{10\left(e^{\frac{16}{5}}-1\right)}$$

10-29) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int e^u \cdot 2du = 2\int e^u du = 2e^u + C$

$u=\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$2du = \frac{dx}{\sqrt{x}}$

$$= \boxed{2e^{\sqrt{x}} + C}$$

(10-36) $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx = \int e^u \cdot (-du) = -e^u + C$

$u = \frac{1}{x}$

$du = \left(\frac{-1}{x^2}\right) dx$

$-du = \frac{1}{x^2}\,dx$

$= \boxed{-e^{\frac{1}{x}} + C}$

(10-37) $\displaystyle\int \left(\frac{e^x + 3x^2}{e^x + x^3 + 1}\right) dx = ?$

$u = (e^x + x^3 + 1)$

$du = (e^x + 3x^2)\,dx$

$= \displaystyle\int \frac{du}{u} = \ln|u| + C$

$= \boxed{\ln\left|e^x + x^3 + 1\right| + C}$

(10-39) $\displaystyle\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} + C = \boxed{\frac{(\ln x)^2}{2} + C} = \boxed{\frac{\ln^2 x}{2} + C}$

$\neq \cancel{\frac{\ln x^2}{2}} + C$

$u = \ln x$

$du = \frac{1}{x}\,dx$

(10-40) $\displaystyle\int \frac{1}{x(2+\ln x)^3}\,dx = ?$

$u = 2 + \ln x$

$\boxed{du = \frac{1}{x}\,dx}$

$= \displaystyle\int \frac{1}{(2+\ln x)^3} \cdot \frac{dx}{x}$

$= \displaystyle\int \frac{1}{u^3}\,du = \int u^{-3}\,du$

$= \frac{u^{-2}}{-2} + C = -\frac{1}{2u^2} + C = \boxed{-\frac{1}{2(2+\ln x)^2} + C}$

(10-47) $\displaystyle\int_{x=0}^{x=1} e^{x^2+x}(2x+1)\,dx = \int_{u=0}^{u=2} e^u \cdot du = e^u\Big|_0^2 = e^2 - e^0 = \boxed{e^2 - 1}$

$u = x^2 + x \Rightarrow \begin{cases} x=1 \Rightarrow u = 1^2+1 = 2 \\ x=0 \Rightarrow u=0 \end{cases}$

$du = (2x+1)\,dx$

10-49) $\int_{t=0}^{t=1/\sqrt{3}} (1+3t^2)^7 \cdot t\,dt$

$\boxed{u = 1+3t^2} \Rightarrow \begin{cases} t = \frac{1}{\sqrt{3}} \Rightarrow u = 1+3\left(\frac{1}{\sqrt{3}}\right)^2 = 2 \\ t=0 \Rightarrow u=1 \end{cases}$

$du = 6t\,dt$

$\boxed{\frac{1}{6}du = t\,dt}$

$$= \int_{u=1}^{u=2} u^7 \cdot \frac{1}{6}du = \frac{1}{6}\int_1^2 u^7 du$$

$$= \frac{1}{6}\frac{u^8}{8}\Big|_1^2 = \boxed{\frac{1}{48}\left[2^8-1\right]}$$

$$= \boxed{\frac{255}{48}}$$

10-45) $\int_1^2 \frac{\ln(t^2)}{t}dt = \int_1^2 \frac{2\ln t}{t}dt = \underbrace{\cancel{2}\frac{\ln^2 t}{\cancel{2}}}_{*39}\Big|_1^2 = (\ln 2)^2 - \underbrace{(\ln 1)^2}_{0} = \boxed{\ln^2 2}$

10-53) $y' = \boxed{\frac{dy}{dx} = -2(x-2)^{-3}+12}$, $\boxed{y(3)=37} \Rightarrow y(x) = ?$ $\int y'dx = y(x)$

$$\boxed{y(x)} = \int\left(-2(x-2)^{-3}+12\right)dx = -2\int u^{-3} + 12x = u^{-2}+12x+C = \boxed{(x-2)^{-2}+12x+C}$$

$u = x-2$
$du = dx$

$$\underbrace{y(3)}_{\boxed{37}} = \underbrace{(3-2)^{-2}}_{1} + \underbrace{12(3)}_{36} + C = \boxed{37+C} \Rightarrow C=0$$

Soln.: $\boxed{y(x) = \frac{1}{(x-2)^2}+12x}$