



ÇANKAYA UNIVERSITY
Department of Mathematics

MATH 111 - Mathematics for Economics and Social Sciences I
2021-2022 Fall

FINAL EXAM
10.01.2022, 10:00

SOLUTIONS

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 100 minutes

Question	Grade	Out of
1		14
2		24
3		19
4		10
5		23
6		21
Total		111

IMPORTANT NOTES:

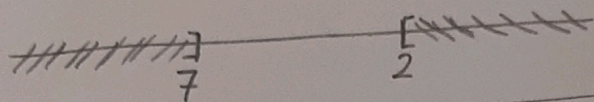
- 1) Please make sure that you have written your student number and name above.
- 2) Check that the exam paper contains 6 questions.
- 3) **SHOW ALL YOUR WORK.** No points will be given to correct answers without reasonable work.

1) a) (7 pts.) Find the solution set of the expression: $|2x + 5| + 1 \geq 10$.
Clearly indicate the solution set.

$$|2x+5| \geq 9 \Rightarrow 2x+5 \geq 9 \quad \text{or} \quad -(2x+5) \geq 9$$

$$2x \geq 4 \quad \quad \quad -2x \geq 9+5=14$$

$$\boxed{x \geq 2} \quad \quad \quad \text{or} \quad 2x \leq -14 \Rightarrow \boxed{x \leq -7}$$



$$\boxed{\text{Solution set: } (-\infty, -7] \cup [2, \infty)}$$

b) (7 pts.) Solve the equation: $\ln(x+1) + \ln(x+7) = \ln(2x-1)$.

$$\ln[(x+1)(x+7)] = \ln(2x-1)$$

$$\Rightarrow x^2 + 8x + 7 = 2x - 1$$

$$\Rightarrow x^2 + 6x + 8 = 0$$

$$(x+4)(x+2) = 0$$

$$\boxed{x = -4 \text{ or } x = -2}$$

$$\begin{array}{lll} \ln(x+1) & \text{defined for } x > -1 \\ \ln(x+7) & \text{" " } x > -7 \\ \ln(2x-1) & \text{" " } x > \frac{1}{2} \end{array}$$

all the terms are defined when $x > \frac{1}{2}$

But $x > \frac{1}{2}$ should be satisfied for all the 3 $\ln(\dots)$ terms to be defined $\Rightarrow$

$$\text{So; } \boxed{\text{s.s.: } \{ \emptyset \}}$$

2) Evaluate the following limits if they exist. Express the solutions clearly. DO NOT USE

L'HOPITAL'S RULE!

(a) (8 pts.) Evaluate $\lim_{x \rightarrow \infty} \left(\frac{3}{x} - \frac{2x^2}{1+x^2} \right)$

$$= \frac{3}{\infty} - \lim_{x \rightarrow \infty} \left[\frac{2x^2}{x^2 \left(\frac{1}{x^2} + 1 \right)} \right] = 0 - 2 = \boxed{-2}$$

0

(b) (8 pts.) $\lim_{x \rightarrow 2} \left(\frac{3x^2 - x - 10}{x^2 + 5x - 14} \right) = \left(\frac{0}{0} \right)$

$$= \lim_{x \rightarrow 2} \left[\frac{(3x+5)(x-2)}{(x+7)(x-2)} \right] = \frac{6+5}{2+7} = \boxed{\frac{11}{9}}$$

(c) (8 pts.) $\lim_{x \rightarrow 4} \left(\frac{x-4}{4 - \sqrt{x+12}} \right) = \left(\frac{0}{0} \right)$

$$= \lim_{x \rightarrow 4} \frac{(x-4)(4 + \sqrt{x+12})}{\underbrace{(4 - \sqrt{x+12})(4 + \sqrt{x+12})}_{16 - (x+12)}} = \lim_{x \rightarrow 4} \frac{\cancel{(x-4)}(4 + \sqrt{x+12})}{-(x-4)}$$

$$= -(4 + \sqrt{4+12}) = \boxed{-8}$$

3) a) (10 pts.) Let the function

$$f(x) = (\ln x)^x$$

be given. Find derivative of the function $f(x)$ at $x = e$. ($f'(e) = ?$)
(Hint: Use logarithmic differentiation)

$$f(x) = (\ln x)^x \Rightarrow \ln f(x) = \ln (\ln x)^x = x \cdot \ln(\ln x)$$

$$\Rightarrow \frac{f'}{f} = (1) \cdot [\ln(\ln x)] + (x) \left[\frac{1}{\ln x} \cdot \frac{1}{x} \right] = \left[\ln(\ln x) + \frac{1}{\ln x} \right]$$

$$\Rightarrow f'(x) = f(x) \cdot \left[\ln(\ln x) + \frac{1}{\ln x} \right] = (\ln x)^x \cdot \left[\ln(\ln x) + \frac{1}{\ln x} \right]$$

$$\Rightarrow \boxed{f'(e)} = \underbrace{(\ln e)}_1 \cdot \left[\underbrace{\ln(\ln e)}_1 + \frac{1}{\ln e} \right] = 1 \cdot [0 + 1] = \boxed{1}$$

$\ln 1 = 0$

b) (9 pts.) Find y' using implicit differentiation, and simplify your answer as much as possible for $\boxed{x^3 y + e^{xy^2} = 5}$.

$$\frac{d}{dx}: x^3 y + e^{xy^2} = 5 \Rightarrow (3x^2 y + x^3 y') + e^{xy^2} \cdot (1 \cdot y^2 + x \cdot 2y \cdot y') = 0$$

$$\Rightarrow y'(x^3 + 2xy \cdot e^{xy^2}) = -3x^2 y - y^2 \cdot e^{xy^2}$$

$$\Rightarrow \boxed{y' = - \frac{3x^2 y + y^2 e^{xy^2}}{x^3 + 2xy e^{xy^2}}}$$

④ (10 pts.) Find real numbers a and b that make $f(x)$ continuous on $\mathbb{R}$ where

$$f(x) = \begin{cases} \frac{16}{x^2} + a, & \text{if } 2 < x, \\ 5, & \text{if } x = 2, \\ 3x - b, & \text{if } x < 2. \end{cases}$$

Explain your answer in detail.

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \left(\frac{16}{x^2} + a \right) = \frac{16}{4} + a = \boxed{4 + a}$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (3x - b) = 3(2) - b = \boxed{6 - b}$$

For continuity at $x=2$:

$$4 + a = f(2) = 5 = 6 - b \quad \left. \vphantom{4 + a = f(2) = 5 = 6 - b} \right\} \Rightarrow \begin{cases} 4 + a = 5 \Rightarrow a = 1 \\ 6 - b = 5 \Rightarrow b = 1 \end{cases}$$

* $f(x)$ is cont. for $x > 2$ and $x < 2$ since f is a polynomial for such x -values

$\Rightarrow f$ is cont. everywhere when $\boxed{a=1, b=1}$

5) Given that $f(x) = e^{-x^2}$

a) (6 pts.) Find the intervals where $f(x)$ is increasing and decreasing.

$$f'(x) = e^{-x^2} \cdot (-2x) = \begin{cases} 0 & \text{when } x=0 \\ \text{d.n.e., no such } x\text{-value} \end{cases}$$

$x=0$ is the only crit. pt. $\Rightarrow$
 $((0, f(0)) = (0, 1))$

x		0	
f'	+	0	-
f	$\nearrow$		$\searrow$

f increasing: $(-\infty, 0)$
 f decreasing: $(0, \infty)$

b) (6 pts.) Find the local extrema of the function $f(x)$.

At $(0, 1)$ there is a local maximum pt.

c) (6 pts.) Find the intervals where $f(x)$ is concave up and concave down.

$$f''(x) = -2[(1) \cdot e^{-x^2} + (x) \cdot e^{-x^2} \cdot (-2x)] = -2e^{-x^2}[1 - 2x^2] = 0$$

x		$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
f''	+		-		+
f	$\cup$		$\cap$		$\cup$

$\Rightarrow x = \pm \frac{1}{\sqrt{2}}$
 f concave up on:
 $(-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \infty)$
 f concave down on:
 $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$(\pm \frac{1}{\sqrt{2}}, e^{-\frac{1}{2}})$: inf. pts

d) (5 pts.) Find the absolute extrema of $f(x)$ on $[-1, 2]$.

x	$f(x)$	
crit. pt. $\leftarrow 0$	1	$\rightarrow$ max. value
end pts. $\left\{ \begin{array}{l} -1 \\ 2 \end{array} \right.$	$\frac{1}{e}$ $e^{-4} = \frac{1}{e^4}$	$\rightarrow$ min. value

* f has abs. max. value of 1 at $x=0$.
 * f has abs. min. value of $\frac{1}{e^4}$ at $x=2$.

6) Evaluate the following integrals.

$$\text{a) (7 pts.) } \int_{-1}^1 (3x^2 - 1)^7 5x dx = \int_{u=2}^{u=2} u^7 \cdot \frac{5}{6} du = \frac{5}{6} \left. \frac{u^8}{8} \right|_2^2 = \boxed{0}$$

$$\boxed{u = 3x^2 - 1}$$

$$du = 6x dx$$

$$\boxed{\frac{1}{6} du = x dx}$$

$$x = -1 \Rightarrow u = 3(-1)^2 - 1 = \boxed{2}$$

$$x = 1 \Rightarrow u = 3(1)^2 - 1 = \boxed{2}$$

(upper limit = lower limit)

$$\text{b) (7 pts.) } \int e^{5x^3} 7x^2 dx = 7 \int e^u \cdot \frac{1}{15} du = \frac{7}{15} e^u + C$$

$$u = 5x^3$$

$$du = 15x^2 dx$$

$$\frac{1}{15} du = x^2 dx$$

$$= \boxed{\frac{7}{15} e^{5x^3} + C}$$

$$\text{c) (7 pts.) } \int_0^1 \frac{x + 2x^3}{1 + x^2 + x^4} dx = \int_1^3 \frac{\frac{1}{2} du}{u} = \frac{1}{2} \ln u \Big|_1^3$$

$$\boxed{u = 1 + x^2 + x^4}$$

$$du = (2x + 4x^3) dx$$

$$= 2(x + 2x^3) dx$$

$$\boxed{\frac{1}{2} du = (x + 2x^3) dx}$$

$$= \frac{1}{2} \ln 3 - \ln 1$$

$$= \boxed{\ln \sqrt{3}}$$

$$\boxed{x = 0 \Rightarrow u = 1}$$

$$\boxed{x = 1 \Rightarrow u = 3}$$