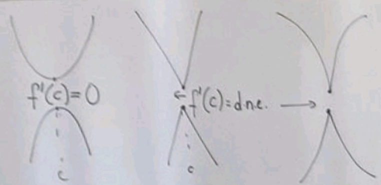
p70: **Theorem:** If the function f is continuous on a closed and bounded interval $[a,b] \Rightarrow$ f has a maximum and minimum value on $[a,b]$

Critical pt.: A number c is called a critical point of f if $f'(c)=0$ or $f'(c)$ does not exist.



* f can have local extremum only at a critical pt.
* " " " abs. " " " " " or at an end pt.

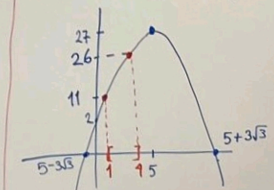
How to find absolute extrema (on an interval $[a,b]$):

- Find the pts. where $f'=0$ } critical pts.
- Find the pts. where f' does not exist. } critical pts.
- Consider such pts. only if they are inside the given interval $[a,b]$.
- Consider end pts. (values of f at $x=a$ & $x=b$).
- Check all the candidates. Both absolute minimum and absolute max. are among them.

Ex.10-1: Find the maximum and minimum values of $\boxed{f(x)=-x^2+10x+2}$ $\left(=-(x^2-10x-2)\right.$ on $[1,4]$.

$$f(x)=-\left[(x-5)^2-25-2\right]=-(x-5)^2+27=0 \Rightarrow (x-5)^2=27 \Rightarrow x-5=\mp\sqrt{27}=\mp 3\sqrt{3}$$
$$x=5\mp 3\sqrt{3}$$

Vertex: $(5,27)$, y-intercept: $(0,2)$



* $f'(x)=-2x+10=0 \Rightarrow \boxed{x=5}$
* $f'(x)=$ d.n.e. $\Rightarrow$ no such x-value

} only critical pt. is $\boxed{x=5}$

$\Rightarrow$ but $x=5 \notin [1,4]$ (crit. pt. is not inside $[1,4]$)

$\Rightarrow$ there is no max/min. value coming from crit. pts.

$\Rightarrow$ Consider end. pts.: $f(1)=11$, $f(4)=26$

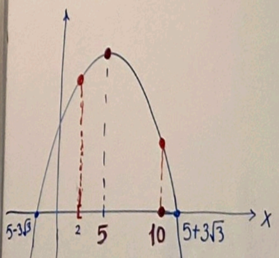
	x	$f(x)$
end pts.	1	11
	4	26

$\Rightarrow$ abs. max. of f on $[1,4]$ is: 26 (at $x=4$)

abs. min. " " " " is: 11 (at $x=1$)

Ex 10.2: $f(x) = -x^2 + 10x + 2$

$[a,b] = [2,10]$



$x=5$ crit. pt. is inside $[2,10] \Rightarrow$

x	f(x)
2	18
5	27 → abs. max
10	2 → abs. min

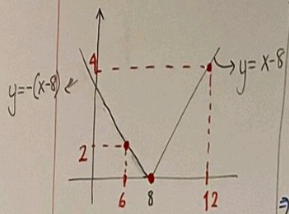
crit. pt. ← 5

end pts. (2 and 10)

Ex 10.3: Find the max. & min. values of

$$f(x) = |x-8| = \begin{cases} x-8, & x \geq 8 \\ -(x-8), & x < 8 \end{cases}$$

on $[6,12]$.



$$f'(x) = \begin{cases} 1, & x > 8 \\ \text{dne}, & x = 8 \\ -1, & x < 8 \end{cases}$$

$\Rightarrow f' = 0 \Rightarrow$ no such x-value

$f' = \text{d.n.e.} \Rightarrow$ at $x=8$

crit. pt. is $x=8$

$f(6) = |6-8| = |-2| = 2$

$f(12) = |12-8| = 4$

$f(8) = 0$

x	f(x)
6	2
8	0 → abs. min. of f
12	4 → abs. max. of f

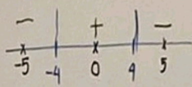
Abs. max. pt.: $(12,4)$

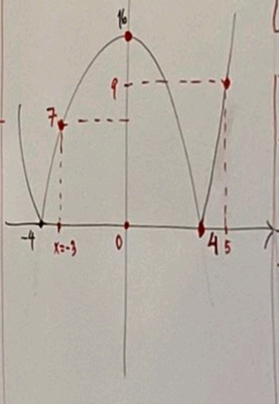
Abs. min. pt.: $(8,0)$

Ex 10.4: $f(x) = |16 - x^2|$, $[a,b] = [-3,5]$

$$f(x) = \begin{cases} -(16-x^2) & x < -4 \\ 16 - x^2 & -4 \leq x \leq 4 \\ -(16-x^2) & x > 4 \end{cases} = \begin{cases} x^2 - 16, & x < -4 \\ 16 - x^2, & -4 \leq x \leq 4 \\ x^2 - 16, & x > 4 \end{cases}$$

$16 - x^2 = (4-x)(4+x)$





x	f(x)
-3	7
0	16 → abs. max
4	0 → abs. min
5	9

end pts. (-3 and 5); crit. pt. (0 and 4)

Abs. max. pt.: $(0,16)$

Abs. min pt.: $(4,0)$

$$f'(x) = \begin{cases} 2x & x < -4 \\ -2x & \boxed{-4 < x < 4} \\ 2x & x > 4 \end{cases}$$

$\Downarrow$

$f'(x)$ d.n.e. at $x = \mp 4$

$f'(x) = 0 \Rightarrow x = 0$

crit. pts: $x=0$, $x = \mp 4$

$\Rightarrow x=0$ & $x=4$ are in the interval $[-3,5]$

p.76/ Find the absolute max. & min. values of $f(x)$ on the given interval.

$\boxed{10\text{-}6)}$ $f(x) = x + \frac{9}{x}$, on $[1,4]$.

$$f'(x) = 1 - \frac{9}{x^2} = \frac{x^2-9}{x^2} = \begin{cases} = 0 \Rightarrow x = \mp 3 \\ = \text{d.n.e.} \Rightarrow x = 0 \end{cases}$$

Crit. pts.: $x=0 \notin [1,4] \rightarrow$ not inside interval X

$x=-3 \notin [1,4] \rightarrow$ " " " X

$\boxed{x=3} \in [1,4]$ ✓ $\Rightarrow f(3) = 3 + \frac{9}{3} = \boxed{6}$

end. pts.: $\boxed{1} \rightarrow f(1) = \boxed{10}$

$\boxed{4} \rightarrow f(4) = 4 + \frac{9}{4} = \boxed{\frac{25}{4}}$

x	f(x)	
1	10	$\rightarrow$ abs. max $\Rightarrow$ Abs. max. pt.: $\boxed{(1,10)}$
3	6	$\rightarrow$ abs. min $\Rightarrow$ Abs. min. pt.: $\boxed{(3,6)}$
4	$\frac{25}{4}$	

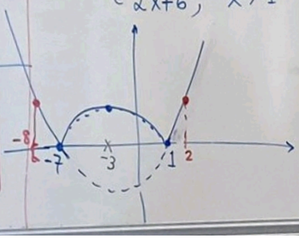
10-9/ $f(x) = |x^2+6x-7|$ on $[-8,2]$

$x^2+6x-7 = (x+7)(x-1)$

+	|	−	|	+
−8	−7	0	1	2

$$f(x) = \begin{cases} x^2+6x-7, & x<-7 \\ -(x^2+6x-7), & -7 \le x \le 1 \\ x^2+6x-7, & x>1 \end{cases}$$

$$f'(x) = \begin{cases} 2x+6, & x<-7 \\ -(2x+6), & -7<x<1 \\ 2x+6, & x>1 \end{cases}$$ at $x=1$ and $x=-7$ f' d.n.e.



$$f'(x) = 0 \Rightarrow \begin{cases} 2x+6=0 \Rightarrow x=-3, \ x<-7 \ \text{X} \\ -(2x+6)=0 \Rightarrow x=-3, \ -7<x<1 \ ✓ \Rightarrow \boxed{x=-3} \text{ crit. pt. ✓} \\ 2x+6=0 \Rightarrow x=-3, \ x>1 \ \text{X} \end{cases}$$

$f'(x) = \text{d.n.e.} \Rightarrow \boxed{x=1 \ \& \ x=-7} \Longrightarrow$ crit. pts. ✓

Crit. pts.: $x=-3$, $x=1$, $x=-7 \in [-8,2]$

end. pts.: $[-8, 2]$

x	f(x)	
−8	9	
−7	0	min.
−3	16	$\rightarrow$ max
1	0	min.
2	9	

* Abs. min. of f is 0 obtained at the interior crit. pts. $x=1$ & $x=-7$ (where $f'(x) = \text{d.n.e.}$)

* Abs. max. of f is 16 obtained at the interior crit. pt. $x=-3$ (where $f'(x)=0$)

10-10 $f(x)=x\sqrt{1-x^2}$, $[-1,1]$

$f(x)=x\cdot(1-x^2)^{1/2}$

$\boxed{f'(x)} = 1\cdot(1-x^2)^{1/2} + x\cdot\frac{(-2x)}{2\sqrt{1-x^2}}$

$=\sqrt{1-x^2}-\frac{2x^2}{2\sqrt{1-x^2}}$

$=\frac{2(1-x^2)-2x^2}{2\sqrt{1-x^2}} = \frac{\cancel{2}[1-2x^2]}{\cancel{2}\sqrt{1-x^2}}$

$=\frac{1-2x^2}{\sqrt{1-x^2}} = \begin{cases} =0 \Rightarrow 1-2x^2=0 \Rightarrow x=\mp\frac{1}{\sqrt{2}}\in[-1,1] \\ =\text{d.n.e} \Rightarrow \underbrace{x=\mp 1}_{\text{also end-pts.}} \in[-1,1] \end{cases}$

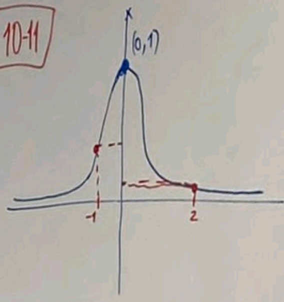
	x	$f(x)$	
end pts. (also crit. pt.)	-1	0	
int. crit. pt.	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	min
int. crit. pt.	$\frac{1}{\sqrt{2}}$	$+\frac{1}{2}$	max
end pts. (also crit. pt.)	1	0	

$f\left(\mp\frac{1}{\sqrt{2}}\right)=\mp\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{1-\frac{1}{2}}\right)=\mp\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}=\mp\frac{1}{2}$

Abs. max. is $\frac{1}{2}$ at interior crit. pt. of type $f'(x)=0$

Abs. min. is $-\frac{1}{2}$ at interior crit. pt. of type $f'(x)=0$

10-11



$\boxed{f(x)=e^{-x^2}}$ → Gaussian distribution

$f(-1)=e^{-(-1)^2}=e^{-1}=\frac{1}{e}$

$f(0)=1$

$f(2)=e^{-(2)^2}=e^{-4}=\frac{1}{e^4}$

$f'(x)=e^{-x^2}\cdot(-2x)=\frac{-2x}{e^{x^2}}=\begin{cases}=0\Rightarrow x=0\in[-1,2]\\ =\text{d.n.e}\Rightarrow \text{no such } x\text{-value}\end{cases}$

x	$f(x)$	
-1	$\frac{1}{e}$	
0	1	→ max ⇒ abs. max. is 1 at int. crit. pt. $x=0$ ($f'(x)=0$)
2	$\frac{1}{e^4}$	→ min ⇒ abs. min. is $\frac{1}{e^4}$ at right end-pt. $x=2$