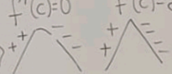
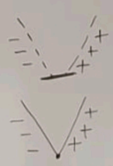
Midterm 2: Dec. 22, Thursday; 17^{30}
Contents: Chs. 6-11
(Ch. 10: absolute/local extrema)

Chapter 11: Curve Sketching:

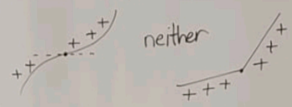
First Derivative Test:

* At a critical pt. $x=c$ either $f'(c)=0$ or $f'(c)=$ d.n.e.
* f has a local extremum at $x=c$ iff f' changes sign at $x=c$:
  * if the sign of f' changes from $+$ to $-$ $\Rightarrow$ 
    $\Rightarrow f$ has a local max. at $x=c$.

* if the sign of f' changes from $-$ to $+$ $\Rightarrow$



then f has a local min. at $x=c$



Ex 11.1: Find the intervals where $\boxed{f(x)=2x^3-9x^2+5}$ is increasing and decreasing and local extrema of this function.

$f'(x)=6x^2-18x=0 \Rightarrow 6x(x-3)=0 \Rightarrow$ crit. pts: $\boxed{x=0,\ x=3}$

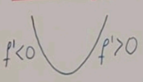
$\boxed{f(0)=5,\ f(3)=-22}$

	$-\infty$ to 0	0 to 3	3 to ∞
x	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
$f'(x)=x(x-3)$	$+$	$-$	$+$
f	↗ incr.	↘ decr.	↗ incr.

$(0,5)$ local max pt.
$(3,-22)$ local min. pt.

Interval(s) of increase: $(-\infty,0)\cup(3,\infty)$
Interval(s) of decrease: $(0,3)$

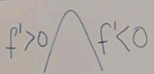
Concavity:



f is concave up if the sign of f' changes from negative to positive

$\Rightarrow (f')' > 0 \Rightarrow f'' > 0$

i.e. f' is increasing $\Rightarrow f \cup$



f is concave down if the sign of f' changes from positive to negative.

$\Rightarrow$ i.e. f' is decreasing;

$(f')' < 0$

Test for concavity:

* if $f''(x) > 0$ then f is concave up at x
* if $f''(x) < 0$ then f is concave down at x

Inflection pt.: An inflection pt. is a point where the concavity changes.

i.e. if;

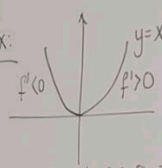
* f is continuous at $x=a$
* $f'' > 0$ on the left of "a" and $f'' < 0$ on the right of $x=a$

or vice-versa.

the $x=a$ is an inflection pt.

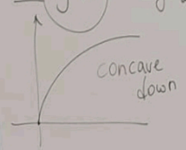
This means; either $f''(a)=0$ or $f''(a)$ does not exist.

Ex:



$y' = 2x$, $y'' = 2 > 0 \Rightarrow y$ is always concave up

Ex: $y=\sqrt{x}$



$y' = \frac{1}{2}x^{-1/2}$, $y'' = -\frac{1}{4}x^{-3/2}$

$y'' = \frac{-1}{4\sqrt{x^3}} < 0$

$\Rightarrow \cap$

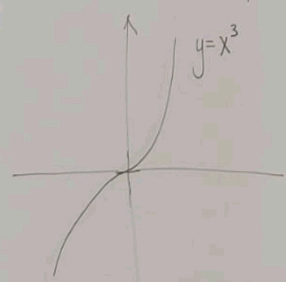
Ex. 11-2: $y = x^3$, $y' = 3x^2 = 0 \Rightarrow x = 0$ crit. pt.

$y'' = 6x = 0 \Rightarrow x = 0$ inflection pt.

x		0	
$f'(x)$	$+$		$+$
$f''(x)$	$-$		$+$

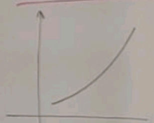
$\to (0,0)$ inflection pt.

f is incr: $(-\infty, \infty)$



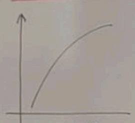
Shape of a graph based on first and second derivatives:

$f'>0,\ f''>0$:



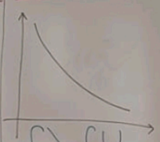
$f \nearrow$, $f \cup$

$f'>0,\ f''<0$:



$f \nearrow$, $f \cap$

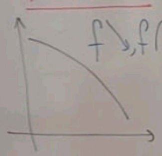
$f'<0,\ f''>0$:



$f \searrow$, $f \cup$

$f'<0,\ f''<0$:

$f \searrow$, $f \cap$



Curve sketching:

- * identify the domain of f, symmetries, x & y-intercepts (if any)
- * first and second derivatives of f
- * Find critical pts. & inflection pts.
- * Make a table and include all the above information.
- * Sketch the curve using the table.

Ex. 11-3: Sketch the graph of $f(x)=x^3+3x^2-24x$

Domain: $(-\infty, \infty)$

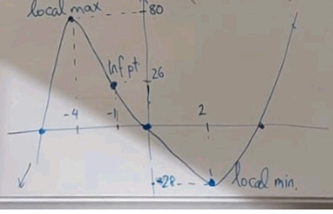
$$\left.\begin{array}{l}\lim_{x\to\infty} f(x)=\infty \ (\text{sign of } x^3 \text{ is } +) \\ \lim_{x\to-\infty} f(x)=-\infty \ (\text{'' '' '' ''})\end{array}\right\}$$

$f(x)=x(x^2+3x-24)$

$x=0 \Rightarrow y=0 \Rightarrow (0,0)$: y-intercept

$y=0 \Rightarrow x=0$ & $x_{2,3}=\dfrac{-3\mp\sqrt{9+96}}{2}=\dfrac{-3\mp\sqrt{105}}{2}$

x-intercepts: $(0,0)$, $\left(\dfrac{-3\mp\sqrt{105}}{2}, 0\right)$



$f'=3x^2+6x-24=3[x^2+2x-8]=3[x+4][x-2]=0$

crit. pts.: $\boxed{x=-4,\ x=2}$

$f(-4)=80,\ f(2)=-28$

$f''=6x+6=6(x+1)=0 \Rightarrow \boxed{x=-1}$ inf. pt.

$\boxed{f(-1)=26}$

x		-4		-1		2	
f'	+	•	−		−	•	+
f''	−		−	•	+		+
f							

-4: local max pt.

-1: inflection pt.

2: local min. pt.

p.83/11.6 $f(x) = x^4 e^{-x}$ ((0,0) both x & y-intercept)

$f'(x) = 4x^3 \cdot e^{-x} + x^4(-e^{-x})$

$= x^3 \cdot e^{-x}(4-x) = 0 \Rightarrow \boxed{x=0,\ x=4}$ crit. pts.

$f(0)=0$

$f(4)=\dfrac{256}{e^4}$

x		0		4	
f'	−	•	+	•	−
f	↘	loc. min	↗	loc. max.	↘

f incr. on: $(0,4)$

f decr. on: $(-\infty,0)\cup(4,\infty)$

f has local min pt.: $(0,0)$

f has local max pt: $\left(4, \dfrac{256}{e^4}\right)$

11.7) $f(x) = \dfrac{\ln x}{x}$

Domain: $(0,\infty)$

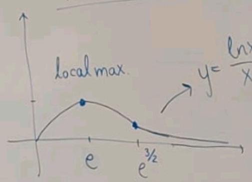
$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (1)(\ln x)}{x^2} = \frac{1-\ln x}{x^2} = \begin{cases} =0 \Rightarrow \ln x = 1 \Rightarrow \boxed{x=e} \\ = \text{d.n.e.} \Rightarrow x = 0 \notin \text{domain} \end{cases}$$

Crit. pt.: $\left(e, \frac{\ln e}{e}\right) = \left(e, \frac{1}{e}\right) \leftarrow$ local max. pt

x	0		e	
f'		+	•	−
f		↗	local max	↘

f incr. on: $(0,e)$

f decr. on: (e,∞)



$$f'' = \frac{\left(-\frac{1}{x}\right)(x^2) - (2x)(1-\ln x)}{(x^2)^2} = \frac{-x-2x+2x\ln x}{x^4}$$

$$= \frac{-3x+2x\ln x}{x^4} = \frac{x(2\ln x - 3)}{x^4} = \frac{2\ln x - 3}{x^3}$$

$$f''=0 \Rightarrow \ln x = \frac{3}{2} \Rightarrow x = e^{\frac{3}{2}} = \sqrt{e^3}$$

x		$e^{3/2}$	
f''	−	•	+
f	∩		∪

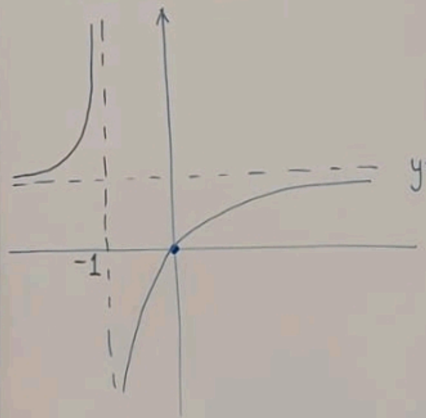
$$\lim_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$$

$$\overset{L.H.}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

11-10) $f(x) = \dfrac{x}{x+1}$

Domain: $(-\infty, -1) \cup (-1, \infty)$

$\lim_{x \to \mp\infty} \left(\dfrac{x}{x+1}\right) = 1$



$f'(x) = \dfrac{(1)(x+1) - (1)(x)}{(x+1)^2} = \dfrac{1}{(x+1)^2} > 0 \quad \forall x$

$\Rightarrow f$ is always increasing ($f \nearrow$ on: $(-\infty, -1) \cup (-1, \infty)$)

f' = d.n.e. when $x = -1 \notin \operatorname{dom} f$

$f' = (x+1)^{-2}$

$f'' = -2(x+1)^{-3}(1) = \dfrac{-2}{(x+1)^3} \begin{cases} > 0 & x < -1 \\ < 0 & x > -1 \end{cases}$

$\Rightarrow f$ concave up when $x < -1$

f // down when $x > -1$