Identify local maxima, minima, and inflection pts.,
intervals of increase - decrease, concavity:

11-12) $\boxed{f(x) = -2x^3 + 21x^2 - 60x}$ Dom f: $\mathbb{R}$

$x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$ both x & y intercept.

$f'(x) = \boxed{-6x^2 + 42x - 60}$
$= -6\left[x^2 - 7x + 10\right] = -6\left[(x-5)(x-2)\right] = 0$

crit. pts.: $\boxed{x = 5,\ x = 2}$

$f(5) = -25$
$f(2) = -52$

$f''(x) = -12x + 42 = -6\left[2x - 7\right] = 0$

$\Rightarrow \boxed{x = \frac{7}{2}}$ inflection pt.

x	$-\infty$ … 0	2	3	$\frac{7}{2}$	4	5	6 … ∞
$f'(x)$	−	•	+		+	•	−
$f''(x)$	+		+	•	−		−
$f(x)$		local min				local max	

trial values
(test values)

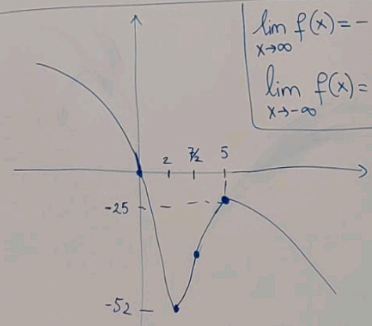
local max.

local min. pt.: $(2, -52)$
local max pt.: $(5, -25)$

$f \nearrow$ f incr. on the interval: $(2, 5)$
f decr. on " " : $(-\infty, 2) \cup (5, \infty)$ $f \searrow$

inflection pt.: $\left(\frac{7}{2}, f\left(\frac{7}{2}\right)\right)$ $\left[-52 < f\left(\frac{7}{2}\right) < -25\right]$

$(f \cup)$ f concave up on: $\left(-\infty, \frac{7}{2}\right)$
$(f \cap)$ f concave down on: $\left(\frac{7}{2}, \infty\right)$



$\lim_{x\to\infty} f(x) = -\infty$

$\lim_{x\to-\infty} f(x) = \infty$

$f(x) = -2x^3 + 21x^2 - 60x = -x\left[2x^2 - 21x + 60\right] = 0$

$-x = 0$, $2x^2 - 21x + 60 = 0$

x-intercepts: $(0,0)$
only x-intercept.

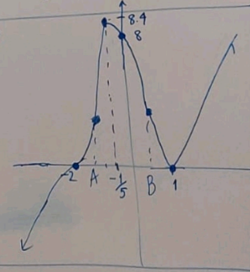
$\Delta = b^2 - 4ac = (-21)^2 - 4(2)(60)$
$= 441 - 480 < 0$
$\Rightarrow$ no other real roots of the above eqn.

11-14) $f(x) = (x-1)^2 \cdot (x+2)^3$

Dom: $(-\infty, \infty)$

y-intercept: $(0, 8)$

x-intercept(s): $(1, 0)$
$(-2, 0)$



$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$

$$\boxed{f'(x)} = [2(x-1)][x+2]^3 + [x-1]^2 \cdot [3(x+2)^2] = 0$$

$\downarrow$ crit. pts.

$$= (x-1)(x+2)^2[2(x+2) + 3(x-1)] = 0$$

$$= \boxed{(x-1)(x+2)^2[5x+1]} = 0$$

crit. pts: $x=1,\ x=-2,\ x=-\frac{1}{5}$

$f(1)=0,\ f(-2)=0,\ f(-\frac{1}{5}) = 8.4$

$$f''(x) = (1)(x+2)^2(5x+1) + (x-1)(2(x+2))(5x+1) + (x-1)(x+2)^2(5)$$

$$= (x+2)\left[\underbrace{(x+2)(5x+1) + 2(x-1)(5x+1) + 5(x-1)(x+2)}_{=0}\right]$$

$$5x^2 + 11x + \cancel{2} + 10x^2 - 8x - \cancel{2} + 5x^2 + 5x - 10$$

$$20x^2 + 8x - 10 = 0$$

$$2(10x^2 + 4x - 5) = 0$$

$$x_{1,2} = \frac{-4 \mp \sqrt{16+200}}{20} = -\frac{1}{5} \mp \frac{\sqrt{216}}{20}$$

f incr. on: $(-\infty, -\frac{1}{5}) \cup (1, \infty)$

$f \searrow$ on: $(-\frac{1}{5}, 1)$

$f \cup$ on: $(-2, A) \cup (B, \infty)$

$f \cap$ on: $(-\infty, -2) \cup (A, B)$

M2/Q2 $\quad f(x) = 2x^2 - x^4$

a) Find the domain of $f(x)$. $\lim_{x\to\infty} f(x) = -\infty \quad \lim_{x\to-\infty} = -\infty$

b) What are the x & y-intercepts of $f(x)$ (if any?)

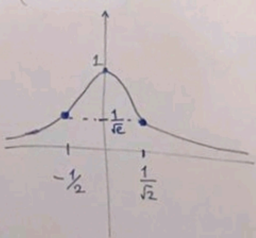
c) Find the intervals of increasing & decr.
Find all local extrema of the function using first derivative test.

d) Inf. pts.; intervals of concave up/down.

e) Absolute extrema on $[a,b]$ $(\in (-\infty, \infty))$

11-17) $f(x)=e^{-x^2}$

* Dom: $(-\infty,\infty)$
* $\lim\limits_{x\to\mp\infty}\left(e^{-x^2}\right)=e^{-\infty}=\frac{1}{\infty}=0$
* $x=0 \Rightarrow f(0)=e^{-0}=1 \Rightarrow (0,1)$: y-int.
* No x-int.; $e^{-x^2}\neq 0$.



$$f'(x)=e^{-x^2}\cdot(-2x)=\frac{-2x}{e^{x^2}}=\begin{cases}=0 \Rightarrow x=0 \rightarrow \text{crit. pt. } (0,1)\\ =\text{d.n.e.} \Rightarrow \text{no such } x\text{-value}\end{cases}$$

x		0	
f'	$+$		$-$
f	$\nearrow$		$\searrow$

f incr. on: $(-\infty,0)$

f decr. on: $(0,\infty)$

x	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1
$f''(x)$	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

$f \cup$ on $\left(-\infty,-\frac{1}{\sqrt{2}}\right)\cup\left(\frac{1}{\sqrt{2}},\infty\right)$

$f \cap$ on : $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

$$f'(x)=-2(x)e^{-x^2}$$

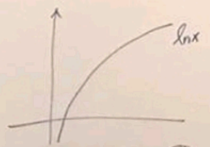
$$f''(x)=-2\left\{(1)(e^{-x^2})+(x)\left[(e^{-x^2})(-2x)\right]\right\}$$

$$=\boxed{-2\underbrace{e^{-x^2}}_{\neq 0}(1-2x^2)}=0$$

$\Rightarrow$ inf. pt.: $1-2x^2=0$

$\left(\mp\frac{1}{\sqrt{2}},\frac{1}{e}\right)$ $\quad x=\mp\frac{1}{\sqrt{2}} \Rightarrow f\left(\mp\frac{1}{\sqrt{2}}\right)=e^{-\left(\mp\frac{1}{\sqrt{2}}\right)^2}=e^{-\frac{1}{2}}=\frac{1}{\sqrt{e}}$

11-19) $f(x)=x\ln|x|=\begin{cases}x\ln x, & x>0\\ x\ln(-x), & x<0\end{cases}$

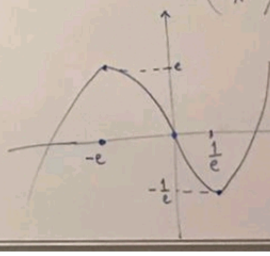


$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}(x\cdot\ln x)=0\cdot(-\infty)$$

$$=\lim_{x\to 0^+}\left(\frac{\ln x}{\frac{1}{x}}\right)\overset{\left(\frac{\infty}{\infty}\right)}{\underset{\text{L.H}}{=}}\lim_{x\to 0}\left(\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right)=\lim_{x\to 0}\left[\left(\frac{1}{x}\right)(-x^2)\right]=0$$

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}\left(x\cdot(\ln(-x))\right)=0\cdot(-\infty)$$

$$=\lim_{x\to 0^-}\left(\frac{\ln(-x)}{\frac{1}{x}}\right)=\frac{-\infty}{-\infty}\overset{\text{L.H.}}{=}\lim_{x\to 0^-}\left(\frac{\frac{-1}{x}}{-\frac{1}{x^2}}\right)=0$$



$$f'(x)=\begin{cases}1.\ln x+x\cdot\frac{1}{x} \checkmark\\ \ln x+1, \quad x>0\\ 1.\ln(-x)+(x)\cdot\frac{-1}{x}\\ \ln(-x)-1, \quad x<0\end{cases}$$

$f'=0 \Rightarrow \begin{cases}\ln x+1=0\\ \ln x=-1\\ x=e^{-1}=\frac{1}{e}>0\\ \ln(-x)-1=0\\ \ln(-x)=1 \Rightarrow -x=e^1 \Rightarrow x=-e\end{cases}$

M2/2021

1.) Find the derivative of the following functions.
Simplify your answers as much as possible.

a.) $f(x) = \ln\left(\frac{x^2+5x+4}{x^3+2}\right)^6$

$f(x) = 6\left[\ln(x^2+5x+4) - \ln(x^3+2)\right]$

$f'(x) = 6\left[\frac{2x+5}{x^2+5x+4} - \frac{3x^2}{x^3+2}\right] = \frac{12x+30}{x^2+5x+4} - \frac{18x^2}{x^3+2}$

$= 6\left[\frac{(2x+5)(x^3+2) - 3x^2(x^2+5x+4)}{(x^3+2)(x^2+5x+4)}\right]$

b.) $f(x) = xe^{-x^2} + 2^{x^2+x} + \log_4\sqrt{1-x}$

$f(x) = xe^{-x^2} + 2^{x^2+x} + \frac{1}{\ln 16}\ln(1-x)$

$f'(x) = \left(1 \cdot e^{-x^2} + x \cdot e^{-x^2}(-2x)\right)$
$+ \left(2^{x^2+x} \cdot (2x+1)(\ln 2)\right)$
$+ \frac{1}{\ln 16}\left(\frac{-1}{1-x}\right)$

$\log_4\sqrt{1-x} = \frac{\ln(1-x)^{1/2}}{\ln 4} = \boxed{\frac{1}{2\ln 4}\ln(1-x)}$

change of base

$\underbrace{2\ln 4}_{\ln 16}$

3-a) implicit: eqn. of tg. line to: $\boxed{2x - 4y^4 + x^2y^6 + 11y^3 = 0 \text{ at } (3,-1)}$

$y-(-1) = m(x-3)$

$\downarrow$

$y'\big|_{(3,-1)} = m$

$2 - 16y^3 \cdot y' + \left[2x \cdot y^6 + x^2 \cdot 6y^5 \cdot y'\right] + 33y^2 \cdot y' = 0$

$y'\left[-16y^3 + 6x^2y^5 + 33y^2\right] = -2 - 2xy^6$

$y' = -\frac{2+2xy^6}{-16y^3+6x^2y^5+33y^2} \Rightarrow y'\big|_{(3,-1)} = \frac{8}{5}$

imp.: $y = y(x)$

tg. line: $y-(-1) = \left(\frac{8}{5}\right)(x-3) \Rightarrow \boxed{8x - 5y = 29}$

$\left[\ln u(x)\right]' = \frac{u'(x)}{u(x)}$

$\left[a^{u(x)}\right]' = a^{u(x)} \cdot u'(x) \cdot \ln a$

$\left(\log_a u(x)\right)' = \frac{u'(x)}{u(x)} \cdot \frac{1}{\ln a}$

b.) $f(x) = \frac{(1+x^3)^7(2+x^5)^8}{(3+x^2)^4} \Rightarrow \ln f(x) = 7\ln(1+x^3) + 8\ln(2+x^5) - 4\ln(3+x^2)$

$\frac{d}{dx}: \frac{f'}{f} = 7\frac{3x^2}{1+x^3} + 8\frac{5x^4}{2+x^5} - 4\frac{2x}{3+x^2}$

$= \boxed{\frac{21x^2}{1+x^3} + \frac{40x^4}{2+x^5} - \frac{8x}{3+x^2}}$

$\Rightarrow f'(x) = \left[f(x)\right]\left[\frac{21x^2}{1+x^3} + \frac{40x^4}{2+x^5} - \frac{8x}{3+x^2}\right]$