Ch 12. Integrals

Antiderivative:

If f is the derivative of F, then F is the anti-derivative of f and

$$\boxed{F'(x) = f(x)}$$

C → constant

$$\Rightarrow \int f(x)\,dx = F(x) + C$$

$\int$ → indefinite integral; $f(x)$ → integrand function; dx → integration variable; C → integration constant

Ex. $\boxed{f(x)=x}$, $F_1(x) = \frac{x^2}{2}$, $F_2(x) = \frac{x^2}{2} + 7$

$F_3(x) = \frac{x^2}{2} - 25, \ldots$

Some basic integrals:

$$\int 1\,dx = x + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$$

$$\int F'(x)\,dx = F(x) + C$$

Ex 12-1. Evaluate the integral

$$\int \left(\frac{1}{x^3} - 2x + 4\right) dx$$

$$= \int x^{-3}\,dx - 2\int x\,dx + 4\int 1\,dx$$

$$= \frac{x^{-2}}{-2} - 2\frac{x^2}{2} + 4 \cdot x + C$$

$$= \boxed{\frac{-1}{2x^2} - x^2 + 4x + C}$$

Ex 12-2, 12-3: HW

Ex 12-4: Find a function

$\boxed{f(x)}$ s.t. $f''(x) = 4 - \frac{8}{x^2}$

and $f(1) = -15$, $f'(1) = 7$.

Soln:

$$f'(x) = \int f''(x)\,dx$$

$$= \int \left(4 - \frac{8}{x^2}\right) dx = \int (4 - 8x^{-2})\,dx$$

$$= 4x - 8\frac{x^{-1}}{-1} + C_1$$

$$\Rightarrow f'(x) = 4x + \frac{8}{x} + C_1$$

$$\underbrace{f'(1)}_{7} = 4(1) + \frac{8}{1} + C_1$$

$$7 = 12 + C_1 \Rightarrow \boxed{C_1 = -5}$$

$$\boxed{f'(x) = 4x + \frac{8}{x} - 5}$$

$$f(x) = \int f'(x)\,dx = \int \left(4x + \frac{8}{x} - 5\right) dx$$

$$\boxed{f(x) = 4\frac{x^2}{2} + 8\ln|x| - 5x + C_2}$$

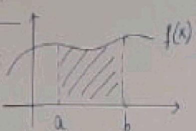
$$\underbrace{f(1)}_{-15} = 2(1)^2 + 8\underbrace{\cancel{\ln|1|}}_{0} - 5(1) + C_2$$

$$-15 - 2 + 5 = C_2 \Rightarrow \boxed{C_2 = -12}$$

$$\Rightarrow \boxed{f(x) = 2x^2 + 8\ln|x| - 5x - 12}$$

Definite Integral:

$$\int_a^b f(x)\,dx$$



f cont. on [a,b]

→ definite integral of f on [a,b] corresponds to the area under f(x) on [a,b].

$$\underbrace{\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)}_{\text{The Fundamental Thm. of Calculus}}$$

Ex 12-5: Evaluate $\int_0^2 (3x^2 + 8x - 5)\,dx = ?$

$$\int_0^2 (3x^2 + 8x - 5)\,dx = \left(3\frac{x^3}{3} + 8\frac{x^2}{2} - 5x\right)\Big|_{x=0}^{x=2}$$

$$= \left[(2)^3 + 4(2)^2 - 5(2)\right] - \underbrace{\left[(0)^3 + 4(0)^2 - 5(0)\right]}_{0}$$

$$= 8 + 16 - 10 = \boxed{14}$$

Ex 12-6: $\int_1^9 \frac{5}{\sqrt{x}}\,dx = 5\int_1^9 x^{-1/2}\,dx$

$$= 5\,\frac{x^{1/2}}{1/2}\Big|_1^9 = 10\left[\sqrt{9} - \sqrt{1}\right] = 10(3-1)$$

$$= 20$$

Ex 12-7: $\int_3^7 \frac{dx}{x} = \ln|x|\Big|_3^7 = \ln 7 - \ln 3 = \boxed{\ln\frac{7}{3}}$

p91 Evaluate the following indefinite integrals:

8) $\int \frac{1}{\sqrt[3]{x^5}}\,dx = \int x^{-5/3}\,dx = \frac{x^{-5/3+1}}{-\frac{5}{3}+1} + C$

$$= \frac{x^{-2/3}}{-\frac{2}{3}} + C = \boxed{-\frac{3}{2\sqrt[3]{x^2}} + C}$$

9) $\int \left(\frac{2u^2 - 3u + 12}{u^2}\right) du = \int \left(\frac{2u^2}{u^2} - \frac{3u}{u^2} + \frac{12}{u^2}\right) du$

$$= 2\int du - 3\int \frac{1}{u}\,du + 12\int u^{-2}\,du$$

$$= 2u - 3\ln|u| + \frac{12u^{-1}}{-1} + C$$

$$= \boxed{2u - 3\ln|u| - \frac{12}{u} + C}$$

Evaluate the following definite integrals:

12-16) $\int_{-1}^{2}(1+4e^{x})dx=\left(x+4e^{x}\right)\Big|_{-1}^{2}$

$=\left(2+4e^{2}\right)-\left((-1)+4e^{-1}\right)$

$=2+4e^{2}+1-\frac{4}{e}$

$=\boxed{3+4e^{2}-\frac{4}{e}}$

12-17) $\int_{1}^{4}5t^{-2}\,dt=?$

$=5\left(\frac{t^{-1}}{-1}\right)\Big|_{1}^{4}=-5\left[\frac{1}{4}-\frac{1}{1}\right]=-5\left(\frac{-3}{4}\right)=\frac{15}{4}$

12-18) $\int_{1}^{9}\left(\frac{1-\sqrt{x}}{\sqrt{x}}\right)dx=\int_{1}^{9}\left(x^{-1/2}-1\right)dx$

$=\left(\frac{x^{1/2}}{1/2}-x\right)\Big|_{1}^{9}=\left(2\sqrt{9}-9\right)-\left(2\sqrt{1}-1\right)$

$=(6-9)-1=\boxed{-4}$

12-20) $\int_{-2}^{-1}\frac{1}{x^{3}}\,dx=\int_{-2}^{-1}x^{-3}\,dx$

$=\frac{x^{-2}}{-2}\Big|_{-2}^{-1}=-\frac{1}{2}\left[\frac{1}{(-1)^{2}}-\frac{1}{(-2)^{2}}\right]$

$=-\frac{1}{2}\left[\frac{1}{x^{2}}\right]_{-2}^{-1}\quad=-\frac{1}{2}\left[1-\frac{1}{4}\right]=\frac{-3}{8}$

Ch 13 Substitution:

Chain rule: $\frac{d}{dx}F(u(x))=\frac{dF(u)}{du}\,\frac{du(x)}{dx}$

$\frac{dF}{dx}$: $F \downarrow u \downarrow x$

$\Rightarrow$ $\boxed{\int f(u(x))\cdot u'(x)\,dx=F(u(x))+C}$

where F is the anti-derivative of $f(x)$.

That is;

$f'(x)=F(x)$.

ex: $\int\frac{2x\,dx}{x^{2}+1}\ \underset{\downarrow}{=}\ \int\frac{du}{u}=\ln|u|+C$

$=\boxed{\ln(x^{2}+1)+C}$

start with the variable "x"

$u=x^{2}+1$
$du=2x\,dx$

substitution

end with the variable "x"

Ex 13.1: Evaluate the integral:

$\int(x^{4}+1)^{2}\cdot\underbrace{4x^{3}\,dx}_{du}$

$\boxed{\begin{array}{l}u=x^{4}+1\\ du=4x^{3}\,dx\end{array}}$

$=\int u^{2}\cdot du=\frac{u^{3}}{3}+C$

$=\frac{(x^{4}+1)^{3}}{3}+C$

$u=x^{4}+1$

Ex 13-2: $\int e^{3x^2} \cdot \underbrace{x\,dx} = ?$

$\boxed{u = 3x^2}$

$du = 6x\,dx$

$\frac{1}{6}\,du = x\,dx$

$\int e^{3x^2} \cdot x\,dx = \int e^{u} \cdot \frac{1}{6}\,du$

$= \frac{1}{6}\int e^{u}\,du = \frac{1}{6}\,e^{u} + C$

$= \boxed{\frac{1}{6}e^{3x^2} + C}$

$\downarrow$
$u = 3x^2$

Ex 13-3: Evaluate the integral:

$\int \underbrace{(x^3+6x^2)}_{u}{}^{7}\,\underbrace{(x^2+4x)\,dx}_{\frac{1}{3}du} = \int u^{7} \cdot \frac{1}{3}\,du$

$u = x^3+6x^2$

$du = (3x^2+12x)\,dx$

$= 3(x^2+4x)\,dx$

$\frac{1}{3}\,du = (x^2+4x)\,dx$

$= \frac{1}{3}\int u^{7}\,du = \frac{1}{3}\,\frac{u^{8}}{8} + C$

$= \boxed{\frac{1}{24}\,(x^3+6x^2)^{8} + C}$

$\downarrow$
$u = x^3+6x^2$

p96 13.6 $\int \frac{3y}{(y^2-2)^4}\,dy = 3\int \frac{\frac{1}{2}\,du}{u^4}$

$\boxed{u = y^2-2}$

$du = 2y\,dy$

$\frac{1}{2}\,du = y\,dy$

$= \frac{3}{2}\int u^{-4}\,du$

$= \frac{\cancel{3}}{2}\left(\frac{u^{-3}}{\cancel{-3}}\right) + C = \frac{1}{2u^3} + C$

$= \boxed{\frac{1}{2(y^2-2)^3} + C}$

13-7) $\int \frac{2x^3+3x}{x^4+3x^2+1}\,dx = \int \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\int \frac{du}{u}$

$u = x^4+3x^2+1$

$du = (4x^3+6x)\,dx$

$du = 2(2x^3+3x)\,dx$

$\boxed{\frac{1}{2}\,du = (2x^3+3x)\,dx}$

$= \frac{1}{2}\ln|u| + C$

$= \boxed{\frac{1}{2}\ln|x^4+3x^2+1| + C}$

13-9) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int e^{u} \cdot 2\,du = 2e^{u} + C$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$= \boxed{2e^{\sqrt{x}} + C}$

13-16) $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx = \int e^{u}(-du) = -\int e^{u}\,du$

$-du$

$u = \frac{1}{x}$

$du = \frac{-1}{x^2}\,dx$

$= -e^{u} + C$

$= \boxed{-e^{1/x} + C}$

13-17) $\int \left(\frac{e^x+3x^2}{e^x+x^3+1}\right)dx = ?$

$u = e^x+x^3+1$

$du = (e^x+3x^2)\,dx$

$= \int \frac{du}{u} = \ln|u| + C$

$= \ln|e^x+x^3+1| + C$