Substitution in Definite Integrals:

$$\int_{x=a}^{x=b} f(\underbrace{u(x)}_{u})\,\underbrace{u'(x)\,dx}_{du} = \int_{u(a)}^{u(b)} f(u)\,du$$

*Don't forget to transform the limits (bounds)!

substitution: $u = u(x)$
$x = a \Rightarrow u = u(a)$
$x = b \Rightarrow u = u(b)$

Ex 13-4: Evaluate the integral:

$$\int_{x=1}^{x=2} \left(\frac{x+2}{\underbrace{x^2+4x+1}_{u}}\right) dx = \int_{u=6}^{u=13} \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\ln|u| \Big|_{u=6}^{u=13}$$

$\boxed{u = x^2+4x+1}$
$du = (2x+4)\,dx$
$= 2(x+2)\,dx$
$\frac{1}{2}du = (x+2)\,dx$

$\Rightarrow$
$x=1 \to u=6$
$x=2 \to u=13$

$= \frac{1}{2}(\ln 13 - \ln 6)$
$= \boxed{\frac{1}{2}\ln\left(\frac{13}{6}\right)}$
$= \boxed{\ln\sqrt{\frac{13}{6}}}$

Ex 13-6: $\int_{-4}^{4} \frac{x}{\sqrt{5-x}}\,dx = ?$

$u = 5-x \Leftrightarrow x = 5-u$
$du = -dx$

$\boxed{u=5-x:}$ $\boxed{x=-4} \Rightarrow \boxed{u} = 5-(-4) = \boxed{9}$
$\boxed{x=4} \Rightarrow \boxed{u} = 5-4 = \boxed{1}$

$$= \int_{u=9}^{u=1} \frac{5-u}{\sqrt{u}}(-du)$$

$$= (-)(-)\int_{u=1}^{u=9} \left(\frac{5}{\sqrt{u}} - \frac{u}{\sqrt{u}}\right) du$$

$$= \int_{1}^{9} \left(5u^{-1/2} - u^{1/2}\right) du = \left(\frac{5u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2}\right)\Big|_1^9$$

$$= 10(\sqrt{9}-\sqrt{1}) - \frac{2}{3}\left[(\sqrt{9})^3 - (\sqrt{1})^3\right]$$

$$= 20 - \frac{2}{3}(27-1) = 20 - \frac{52}{3} = \boxed{\frac{8}{3}}$$

Ex 13-7: $\int_0^2 8e^{\frac{x^4}{5}} \cdot \underbrace{x^3\,dx}_{} = ?$

$\boxed{u = \frac{x^4}{5}} \Rightarrow du = \frac{4x^3}{5}\,dx$

$\frac{5}{4}du = x^3\,dx$

$x=0 \Rightarrow u=0$
$x=2 \Rightarrow u = \frac{2^4}{5} = \frac{16}{5}$

$$= \int_{u=0}^{u=\frac{16}{5}} \cancel{8}^{2}\, e^u \cdot \left(\frac{5}{\cancel{4}_1}du\right)$$

$$= 10\,e^u \Big|_0^{\frac{16}{5}} = 10\left(e^{\frac{16}{5}} - \underbrace{e^0}_{1}\right)$$

$$= \boxed{10\left(e^{16/5} - 1\right)}$$

p97 | 13/25) $\int_1^2 \frac{\ln t^2}{t}\,dt = 2\int_1^2 \underbrace{\frac{\ln t}{t}\,dt}_{du}$

$\underbrace{u = \ln t}_{} \Rightarrow \boxed{du = \frac{1}{t}dt}$

$t=1 \Rightarrow u = \ln 1 = 0$
$t=2 \Rightarrow u = \ln 2$

$$= 2\int_{u=0}^{u=\ln 2} u \cdot du = \cancel{2}\,\frac{u^2}{\cancel{2}}\Big|_{u=0}^{u=\ln 2}$$

$$= (\ln 2)^2 - \cancel{(0)^2}$$

$$= \boxed{(\ln 2)^2 = \ln^2 2}$$

$$\neq \ln 2^2 = \ln 4$$

13/27

$$\int_{x=0}^{x=1} e^{\overbrace{x^2+x}^{u}} \cdot \underbrace{(2x+1)\,dx}_{du} = \int_0^2 e^u\,du = e^u \Big|_0^2$$

$$= e^2 - e^0 = \boxed{e^2-1}$$

$u = x^2 + x \Rightarrow du = (2x+1)\,dx$

$x=0 \Rightarrow u=0$

$x=1 \Rightarrow u=2$

13-30

$$\int_{x=1}^{x=e^5} \left(\frac{\overbrace{1+2\ln x}^{u}}{x}\right) dx$$

$\frac{1}{2}du$

$u = 1 + 2\ln x \Rightarrow du = 2 \cdot \frac{1}{x}\,dx$

$\frac{1}{2}du = \frac{dx}{x}$

$x=1 \Rightarrow u = 1 + 2\cancel{\ln 1}^{0} = 1$

$x = e^5 \rightarrow u = 1 + 2\underbrace{\ln e^5}_{5} = 1 + 2(5) = 11$

$$= \frac{1}{2}\int_{u=1}^{u=11} u\,du = \frac{1}{2}\,\frac{u^2}{2}\Big|_1^{11} = \frac{1}{4}\left(11^2 - 1^2\right) = \frac{1}{4}(121-1) = \boxed{30}$$

Fall 2021 / Final, Q6 / Evaluate the following integrals:

a) $$\int_{-1}^{1} (\underbrace{3x^2-1}_{u})^7 \cdot 5\,\underbrace{x\,dx} = \int_{u=2}^{u=2} u^7 \cdot 5\left(\frac{1}{6}du\right) = 0$$

upper limit = lower limit

$\boxed{u = 3x^2-1} \Rightarrow du = 6x\,dx$

$\frac{1}{6}du = x\,dx$

$\Downarrow$

$x=-1 \Rightarrow u = 3(-1)^2 - 1 = 2$

$x = 1 \Rightarrow u = 3(1)^2 - 1 = 2$

$$= \frac{5}{6}\,\frac{u^8}{8}\Big|_2^2 = 0$$

b.) $$\int e^{5x^3} \cdot 7\underbrace{x^2\,dx}_{\frac{1}{15}du} = ?$$

$u = 5x^3 \Rightarrow du = 15x^2\,dx$

$\frac{1}{15}du = x^2\,dx$

$$= \frac{7}{15}\int e^u\,du = \frac{7}{15}e^u + C$$

$$= \boxed{\frac{7}{15}e^{5x^3} + C}$$

c.) $$\int_0^1 \frac{x+2x^3}{1+x^2+x^4}\,dx$$

$u = 1 + x^2 + x^4 \Rightarrow du = (2x + 4x^3)\,dx$

$\frac{1}{2}du = (x + 2x^3)\,dx$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=3$

$$= \int_{u=1}^{u=3} \frac{\frac{1}{2}du}{u} = \frac{1}{2}\ln u \Big|_1^3$$

$$= \frac{1}{2}\ln 3 - \cancel{\ln 1}^{0} = \boxed{\frac{\ln 3}{2}}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$