Ch 8 Implicit Differentiation:

$y = y(x)$

- $x^2 + y^2 = 1$
- $ye^{y} + 2x - \ln y = 0$
- $e^x + e^y = \sqrt{x+2y}$

} define y as a function of x → implicitly

$\Rightarrow \dfrac{dy}{dx} = y'(x) \rightarrow ?$

* When we find $y'(x)$ without solving y explicitly as a function of x ⇒ this process is called implicit differentiation

Ex 8.1: Find y' using the equation

$$\boxed{y + y^3 = 3x^2 + 1}$$

* keep in mind that $y = y(x)$

$$\frac{d}{dx}(y) + \boxed{\frac{d}{dx}(y^3)} = \frac{d}{dx}(3x^2+1) \qquad y \to x$$

$$\underbrace{\frac{dy}{dx}}_{y'} + (3y^2)\underbrace{\frac{dy}{dx}}_{y'} = 6x + 0$$

$$y'\left[1 + 3y^2\right] = 6x$$

$$\Rightarrow \boxed{y' = \frac{6x}{1+3y^2}}$$

$$\frac{d}{dx}(y^n) = n\,y^{n-1}\cdot\frac{dy}{dx}$$

$y = y(x)$

tg. line eqn:

$$y - y_0 = m(x - x_0)$$

$$y - \sqrt{3} = \left(-\frac{1}{\sqrt{3}}\right)(x-1)$$

$$y = -\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}} + \sqrt{3}$$

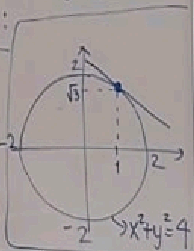
$$\boxed{y = -\frac{1}{\sqrt{3}}x + \frac{4}{\sqrt{3}}}$$

Ex 8.2: Find the slope $(m = y'(x_0))$ of the tangent line to the curve $\boxed{x^2+y^2=4}$ at the point $(\underset{x_0}{1}, \underset{y_0}{\sqrt{3}})$.

using implicit diff.:

$$2x + 2y\cdot y' = 0$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$



$$\underbrace{y'(\underset{x_0}{1})}_{m = \text{slope of tg. line}} = -\left(\frac{x}{y}\right)\Bigg|_{(x_0,y_0)} = \boxed{-\frac{1}{\sqrt{3}}}$$

Steps of implicit diff are:

* differentiate everything w.r.t. "x"
* solve for y'

Ex. 8.4: Find y' using implicit diff. where $y = y(x) \to y' = ?$

$$\boxed{\underbrace{x^2 e^y}_{\text{product}} + y = e^{3x}}$$

$$\underbrace{\left[(2x)(e^y) + (x^2)(e^y \cdot y')\right]} + [y'] = (e^{3x})(3)$$

$$y'\left[x^2 \cdot e^y + 1\right] = 3e^{3x} - 2xe^y$$

$$\Rightarrow \boxed{y' = \frac{3e^{3x} - 2xe^y}{x^2 e^y + 1}}$$

Ex 8.6: $y' = ?$

$$\underbrace{ye^{xy}}_{\text{product}} + \underbrace{x^4 \ln x}_{\text{product}} = e^{3x}$$

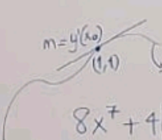
$$\left\{(y' \cdot e^{xy}) + (y)\left[e^{xy} \cdot \underbrace{\frac{d}{dx}(xy)}_{(1 \cdot y + x \cdot y')}\right]\right\} + \left\{(4x^3)(\ln x) + \underbrace{(x^4) \cdot \left(\frac{1}{x}\right)}_{x^3}\right\} = 3e^{3x}$$

$$\Rightarrow y'e^{xy} + e^{xy}\left[y^2 + xy \cdot y'\right] + (4x^3)(\ln x) + x^3 = 3e^{3x}$$

$$y'\underbrace{\left[e^{xy} + e^{xy} \cdot xy\right]}_{e^{xy}[1+xy]} = 3e^{3x} - 4x^3 \ln x - x^3 - y^2 \cdot e^{xy}$$

$$\Rightarrow \boxed{y' = \frac{3e^{3x} - 4x^3 \ln x - x^3 - y^2 \cdot e^{xy}}{e^{xy}[1+xy]}}$$

Ex 8.7: Find the $\overbrace{\text{slope of the tangent line}}^{y'(x_0) = y'(1)}$ to the curve $\boxed{x^8 + 4x^2y^2 + y^8 = 6}$ at the point $(1, 1)$, with $1 \to x_0$, $1 \to y_0$.



$$x^8 + 4x^2y^2 + y^8 = 6$$

$$8x^7 + 4\left[2x \cdot y^2 + x^2 \cdot 2y \cdot y'\right] + 8y^7 \cdot y' = 0$$

$$y'\left[8x^2y + 8y^7\right] = -8x^7 - 8xy^2$$

$$y' = -\frac{8x^7 + 8xy^2}{8y^7 + 8x^2y} = -\frac{8(x^7 + xy^2)}{8(y^7 + x^2y)} = \boxed{-\frac{x^7 + xy^2}{y^7 + yx^2}}$$

$$y'(1) = -\left.\frac{x^7 + xy^7}{y^7 + yx^2}\right|_{(1,1)} = -\frac{1+1}{1+1} = -\frac{2}{2} = -1$$

HW problems: (Redo)
Examples 8.3, 8.5, 8.8

Exercises on page 60:

Find y' using implicit differentiation:

8.2 $\underbrace{xye^x}_{\text{triple product}} + \underbrace{(x+2y)^2}_{\text{chain rule}} = x \qquad y=y(x) \Rightarrow y' = ?$

$$\left[1(ye^x) + (x)(y')(e^x) + (xy)(e^x)\right] + 2(x+2y)^1 \cdot (1+2y') = 1$$

$$y'\left[xe^x + 4(x+2y)\right] = 1 - ye^x - xye^x - 2(x+2y)$$

$$\boxed{y' = \frac{1 - ye^x(1+x) - 2(x+2y)}{xe^x + 4(x+2y)}}$$

8.4: $x = y + y^{2/3}$:

$$1 = y' + \frac{2}{3}(y^{-1/3}) \cdot y'$$

$$1 = y'\left[1 + \frac{2}{3\sqrt[3]{y}}\right] = y'\left[\frac{3\sqrt[3]{y}+2}{3\sqrt[3]{y}}\right]$$

$$\Rightarrow \boxed{y' = \left(\frac{3\sqrt[3]{y}}{3\sqrt[3]{y}+2}\right)}$$

8.5 $(1+e^{-x})^2 = \ln(x+y)$

$$2(1+e^{-x})\left[-e^{-x}\right] = \frac{1}{x+y}(1+y')$$

$$-2e^{-x}(1+e^{-x}) - \frac{1}{x+y} = \frac{y'}{x+y}$$

$$-2e^{-x}(1+e^{-x})(x+y) - 1 = y'$$

$$\boxed{y' = -2(x+y)e^{-x}(1+e^{-x}) - 1}$$

8.6: $\ln y = y^3 + \ln x$

$$\frac{1}{y} \cdot y' = 3y^2 \cdot y' + \frac{1}{x}$$

$$y'\left[\frac{1}{y} - 3y^2\right] = \frac{1}{x}$$

$$y'\left(\frac{1-3y^3}{y}\right) = \frac{1}{x}$$

$$\boxed{y' = \frac{y}{x(1-3y^3)}}$$

8.11: $xy^2 = 1 + \ln(xy)$

$$\boxed{(1 \cdot y^2 + x \cdot 2y\, y')} = 0 + \frac{1}{xy}\left[1 \cdot y + x\, y'\right] = \frac{y}{xy} + \frac{x}{xy}y' = \boxed{\frac{1}{x} + \frac{1}{y}y'}$$

$$y'\left[2xy - \frac{1}{y}\right] = \frac{1}{x} - y^2$$

$$y'\left(\frac{2xy^2-1}{y}\right) = \frac{1-xy^2}{x} \Rightarrow y' = \left(\frac{1-xy^2}{x}\right)\left(\frac{y}{2xy^2-1}\right)$$

$$\Rightarrow \boxed{y' = \frac{y(1-xy^2)}{x(2xy^2-1)}}$$

8.12: $e^y + x^2e^x = 18$:

$$\Rightarrow e^y \cdot (y') + (2x \cdot e^x + x^2 \cdot e^x) = 0$$

$$y' = -\frac{(2xe^x + x^2e^x)}{e^y}$$

$$\boxed{y' = -\frac{xe^x(2+x)}{e^y}}$$

$$\Rightarrow \boxed{y' = -(2x+x^2) \cdot e^{x-y}}$$

8.13: $\underbrace{y^2 \ln y}_{\text{product}} = \underbrace{x^3 \cdot e^x}_{\text{product}}$

$$\left[(2y \cdot y')(\ln y) + (\cancel{y^2}^{\,y})\left(\frac{1}{\cancel{y}} \cdot y'\right)\right] = (3x^2)(e^x) + (x^3)(e^x)$$

$$y'\underbrace{[2y \cdot \ln y + y]}_{y[2\ln y + 1]} = x^2 \cdot e^x [3 + x]$$

$$\Rightarrow \boxed{y' = \frac{x^2 \cdot (3+x)e^x}{y[2\ln y + 1]}}$$

8.15: $\frac{2}{x} + \frac{7}{y} = 9 \Rightarrow \left(\frac{-2}{x^2}\right) + \left(\frac{-7}{y^2}\right)(y') = 0$

$y' = ?$

$$\left(-\frac{7}{y^2}\right)y' = \frac{2}{x^2} \Rightarrow \boxed{y' = -\frac{2y^2}{7x^2}}$$

8.18 $\boxed{3x - 2y + 8x^2 + 5y^2 + 9e^{9x} + 7e^{2y} = 16} \Rightarrow y'\Big|_{(0,0)} = ?$

$$3 - 2y' + 16x + 10y\,y' + 81e^{9x} + 14e^{2y}\,y' = 0$$

$$y'[-2 + 10y + 14e^{2y}] = -3 - 16x - 81e^{9x}$$

$$\Rightarrow \boxed{y' = -\frac{3 + 16x + 81e^{9x}}{-2 + 10y + 14e^{2y}}}$$

8.22 $\underbrace{\sqrt{11+y^2}}_{(11+y^2)^{1/2}} - 12(xy) + 2y^2 + 4x = 0$ at $(1,5)$

$y'\Big|_{(1,5)} = ?$

$$\frac{1}{\cancel{2}}(11+y^2)^{-1/2}(\cancel{2}y \cdot y') - 12[1 \cdot y + x \cdot y'] + 4y \cdot y' + 4 = 0$$

$$\left(\frac{y}{\sqrt{11+y^2}}\right)y' - 12x \cdot y' + 4y\,y' = 12y - 4$$

$$y'\underbrace{\left[\frac{y}{\sqrt{11+y^2}} - 12x + 4y\right]}_{} = 12y - 4 \Rightarrow$$

$$y'\left(\frac{y + (4y - 12x)\sqrt{11+y^2}}{\sqrt{11+y^2}}\right) = 12y - 4$$

$$\Rightarrow \boxed{y' = \frac{(12y-4)(\sqrt{11+y^2})}{y + (4y-12x)\sqrt{11+y^2}}}$$

$$y'\Big|_{(1,5)} = \left(\frac{(12(5)-4)\sqrt{11+25}}{5 + (4(5) - 12(1))\sqrt{11+25}}\right) = \frac{(56)(6)}{5 + 8(6)} = \frac{336}{53}$$

$(1,5)$: x_0, y_0

8.24 $y'\Big|_{(2,3)} = ?$ $\quad \ln(xy) + xy^2 - \ln(3x) - 6y = 0$

$$\frac{1}{xy}[1 \cdot y + x\,y'] + (1\,y^2 + x \cdot 2y\,y') - \frac{1}{\cancel{3}x}(\cancel{3}) - 6y' = 0$$

$$y'\left[\frac{x}{xy} + 2xy - 6\right] = \frac{1}{\cancel{x}} - \frac{\cancel{y}}{xy} - y^2 \Rightarrow y'\left[\frac{1}{y} + 2xy - 6\right] = -y^2$$

$$\boxed{y' = \frac{-y^2}{1 + 2xy^2 - 6}\,y}$$

$$\Rightarrow y' = -\frac{y^3}{1 + 2xy^2 - 6} \Rightarrow y'\Big|_{(2,3)} = -\frac{27}{1 + 36 - 6} = \boxed{-\frac{27}{31}}$$