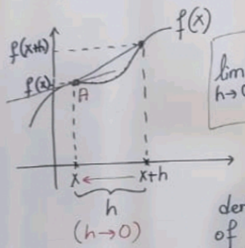
Ch6. Derivatives:



$$\boxed{\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{(x+h)-(x)}\right)} = \text{slope of the tangent line drawn to the graph of } y=f(x) \text{ at } \underbrace{(x, f(x))}_{A}$$

$$\| $$

$$\underbrace{f'(x)}$$

derivative of $y=f(x)$ at $(x, f(x))$.

$f'(x)$ can also be equivalently described as:

$$f'(x) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)$$

We can think of the derivative as:

i) the rate of change of a function f

or

ii) the slope of the tangent line drawn to $y=f(x)$

Notations: for $y=f(x)$

⊛ y', $f'(x)$, $\frac{dy}{dx}$, $\frac{d}{dx}f(x)$ → denotes derivatives; and

⊛ $f'(a)$, $\left.\frac{dy}{dx}\right|_{x=a}$ to denote the value of $f(x)$ at a certain pt.

* Note that; the derivative $f'(x)$ is a function, where as $f'(a)$ is a real number.

Higher-order derivatives:

$\frac{d}{dx}(f'(x)) = f''(x)$ → second order derivative of $f(x)$

equivalent notations: y'', $f''(x)$, $\frac{d^2y}{dx^2}$ $(y=f(x))$

In general; $\frac{d^3y}{dx^3} = f'''(x)$, ...

$\frac{d^4y}{dx^4} = f^{(4)}(x)$ . . . . → $\frac{d^nf}{dx^n} = f^{(n)}(x)$

Ex. 6-1: $f(x) = 7x^3 - 18x$

$$\begin{cases} f'(x) = 21x^2 - 18 \\ f''(x) = 42x \\ f'''(x) = 42 \\ f^{(IV)}(x) = 0 \end{cases}$$

$$\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$$

$$\boxed{x^0 = 1}$$

Differentiation Formulas:

Using the defn. of derivative:

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

we obtain: ($c \to$ constant)

* $\dfrac{dc}{dx} = 0$ $\quad \left(f(x) = f(x+h) = c \right.$ $\left. f(x) = c \right)$

* $f(x) = x$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)-(x)}{h} = \lim_{h\to 0} (1) = \boxed{1}$$

* $f(x) = x^n$ $\quad (n \in \mathbb{Z}_+)$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{(x+h)-(x)} = \lim_{h\to 0} \frac{\cancel{[(x+h)-(x)]}\overbrace{[(x+h)^{n-1}+\cdots+x^{n-1}]}^{n\text{-terms}}}{\cancel{[(x+h)-x]}}$$

power rule $\quad = \boxed{n x^{n-1}}$

* $f(x) = \sqrt{x}$:

$$f(x) = x^{1/2} \Rightarrow f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-1/2} = \boxed{\frac{1}{2\sqrt{x}}}$$

* $f \to$ function, $c \to$ constant

$$(cf)' = \frac{d}{dx}(c f(x)) = c f'(x)$$

* $f, g \to$ function

$$(f \mp g)' = f' \mp g'$$

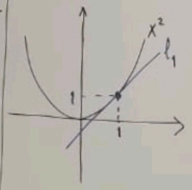
$\boxed{\text{p. }49/6\text{-}7}$ $\quad f(x) = \dfrac{x^3 - x}{\sqrt{x}} \Rightarrow f'(x) = ?$

$$\Rightarrow f(x) = \frac{x^3}{x^{1/2}} - \frac{x}{x^{1/2}} = \boxed{x^{5/2} - x^{1/2}}$$

$$f'(x) = \left(\frac{5}{2}\right) x^{\left(\frac{5}{2}-1\right)} - \left(\frac{1}{2}\right) x^{\left(\frac{1}{2}-1\right)} = \frac{5}{2} x^{\frac{3}{2}} - \frac{1}{2} x^{-1/2}$$

$$= \boxed{\frac{5}{2}(\sqrt{x})^3 - \frac{1}{2\sqrt{x}}}$$

Ex. 6-3: Find the equation of the tangent line (ℓ_1) to the graph of $f(x) = x^2$ at the point $(1,1)$.



slope of $\ell_1 = m_1 = f'(1)$

$$f'(x) = 2x \Rightarrow f'(1) = 2(1) = \underset{m_1}{\underbrace{2}} \rightarrow \text{slope of tangent line}$$

$\Rightarrow$ tangent line eqn.:

$$y - 1 = 2(x-1) \Rightarrow y - 1 = 2x - 2$$

$$y = 2x - 2 + 1 = 2x - 1$$

$$\Rightarrow \boxed{y = 2x - 1} \text{ eqn. of tg. line.}$$

Differentiation Rules:

Product rule: $\boxed{(f\cdot g)'(x) = f'(x)\cdot g(x) + g'(x)\cdot f(x)}$

Quotient rule: $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - g'f}{g^2}$

Reciprocal rule: $\boxed{\left(\dfrac{1}{f}\right)'} = \dfrac{\overset{0}{(1)'}\cdot f - f'(1)}{f^2}$

$$= \boxed{-\frac{f'}{f^2}}$$

$$\lim_{h\to 0}\left[\frac{(f\cdot g)(x+h) - (f\cdot g)(x)}{(x+h) - x}\right] = (f\cdot g)'(x)$$

$\boxed{p49)\ 6.9}$ $f(x) = (x^2+2)(x^2-3) \Rightarrow f'(x) = ?$

Method 1: (power rule) $f(x) = x^4 - x^2 - 6$

$\boxed{f'(x) = 4x^3 - 2x}$

Method 2: (product rule) $f'(x) = \underbrace{(2x)}_{\frac{d}{dx}(x^2+2)}(x^2-3) + (x^2+2)\underbrace{(2x)}_{\frac{d}{dx}(x^2-3)}$

$$\Rightarrow f'(x) = 2x^3 - 6x + 2x^3 + 4x = \boxed{4x^3 - 2x}$$

6-11) $f(x) = \dfrac{x^2+12}{5x-2} \Rightarrow$ using quotient rule

we have:

$$f'(x) = \frac{\left[\frac{d}{dx}(x^2+12)\right]\cdot(5x-2) - (x^2+12)\cdot\left[\frac{d}{dx}(5x-2)\right]}{(5x-2)^2}$$

$$= \frac{(2x)(5x-2) - (x^2+12)(5)}{(5x-2)^2} = \frac{10x^2 - 4x - 5x^2 - 60}{(5x-2)^2} = \boxed{\frac{5x^2 - 4x - 60}{(5x-2)^2}}$$

Exponential and Logarithmic Function Derivatives:

$$\boxed{\frac{d}{dx}(e^x) = e^x, \quad \frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Ex 6-12: Find the derivative of: $f(x) = \dfrac{x^4}{e^x - x^2}$

Using quotient rule:

$$f'(x) = \frac{(4x^3)(e^x - x^2) - (e^x - 2x)(x^4)}{(e^x - x^2)^2}$$

$$= \frac{4x^3e^x - 4x^5 - x^4\cdot e^x + 2x^5}{(e^x - x^2)^2}$$

$$= \boxed{\frac{4x^3e^x - x^4\cdot e^x - 2x^5}{(e^x - x^2)^2}}$$

Ex 6-13: Find the derivative of:

$$f(x) = \frac{1}{x - e^x + \ln x}$$

Using reciprocal rule: $\left(\frac{1}{g(x)}\right)' = \boxed{\frac{-g'(x)}{[g(x)]^2}}$

$$f'(x) = -\frac{\frac{d}{dx}(x - e^x + \ln x)}{(x - e^x + \ln x)^2} = -\frac{1 - e^x + \frac{1}{x}}{(x - e^x + \ln x)^2}$$

$$= \boxed{\frac{-(x - xe^x + 1)}{x \cdot (x - e^x + \ln x)^2}}$$

6-14) $f(x) = x^2 \underbrace{\ln(x^3)}_{3\ln x} = 3(x^2)(\ln x)$

$$\underset{\text{product rule}}{f'(x)} = 3\left[(2x)(\ln x) + (x^2)\cdot\left(\frac{1}{x}\right)\right] = \boxed{(6x)(\ln x) + 3x}$$

$$= \boxed{x\,[6\ln x + 3]}$$

6-19) $f(x) = x^4 \cdot e^x \cdot \ln x$

$$f(x) = x^4 \cdot (e^x \cdot \ln x)$$

$$f'(x) = (4x^3)[e^x \cdot \ln x] + (x^4)\left[\frac{d}{dx}(e^x) \cdot \ln x + (e^x)\frac{d}{dx}(\ln x)\right]$$

$$= (4x^3)(e^x \ln x) + (x^4)(e^x \ln x) + (x^4)(e^x)\cdot\left(\frac{1}{x}\right)$$

$$= (x^3 e^x \ln x)[4 + x] + x^3 \cdot e^x$$

$$= x^3 \cdot e^x\left[(4+x)\ln x + 1\right]$$

6-22) $f(x) = \frac{1}{\ln(4x)} \Rightarrow f'(x) = ?$

$\ln(4x) = \ln 4 + \ln x$

$$f'(x) = \frac{-\frac{d}{dx}(\ln 4 + \ln x)}{[\ln(4x)]^2} = -\frac{\left(0 + \frac{1}{x}\right)}{\ln^2(4x)} = \frac{-1}{x\,\ln^2(4x)}$$

$[\ln(4x)]^2 \neq \ln(4x^2)$; $\ln^2(4x) \neq 2\ln(4x)$

6-23) $f(x) = e^x \cdot e^x \cdot e^x \Rightarrow f'(x) = ?$ $\left(\text{i.e. } \frac{d}{dx}(e^{3x}) = ?\right)$

$$f'(x) = (e^x)(e^x \cdot e^x) + (e^x)(e^x)(e^x) + (e^x)(e^x)(e^x) = 3e^x \cdot e^x \cdot e^x = \boxed{3e^{3x}}$$

6-24) $f(x) = \frac{2 - 3\ln x}{5\ln x + 1} \Rightarrow f'(x) = \frac{\left[\frac{d}{dx}(2 - 3\ln x)\right][5\ln x + 1] - \left[\frac{d}{dx}(5\ln x + 1)\right][2 - 3\ln x]}{(5\ln x + 1)^2}$

$$\Rightarrow f'(x) = \frac{\left(-3\frac{1}{x}\right)(5\ln x + 1) - \left(5\frac{1}{x}\right)(2 - 3\ln x)}{(5\ln x + 1)^2} = \frac{\cancel{-15\ln x} - 3 - 10 + \cancel{15\ln x}}{x\,(5\ln x + 1)^2}$$

$$= \frac{\frac{-3(5\ln x + 1)}{x} - \frac{5(2 - 3\ln x)}{x}}{(5\ln x + 1)^2} = \boxed{\frac{-13}{x\,(5\ln x + 1)^2}}$$