Chain Rule:

If f&g are differentiable $\Rightarrow$

$\underbrace{f(g(x))}_{(f\circ g)(x)}$ is also differentiable, &

$$\boxed{\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)}$$

NOT!!; $(f(g(x)))' \neq f'(g'(x))$

or

$y = f \rightarrow u = g \rightarrow x$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Ex. 7.1: Find $\frac{d}{dx}(3x^2+1)^5 = ?$

$$\boxed{\begin{aligned} f(u) &= u^5, & u(x) &= 3x^2+1 \\ f'(u) &= 5u^4, & u'(x) &= 6x \end{aligned}}$$

$$\boxed{\frac{d}{dx}(f(u(x)))} = f'(u(x)) \cdot u'(x)$$

$$= 5(3x^2+1)^4 \cdot (6x)$$

$$= \boxed{30x(3x^2+1)^4}$$

Ex. 7.2: Find $f'(x)$ where $f(x) = e^{x^5}$

$f(u) = e^u$, $g(x) = x^5$

$f'(u) = e^u$, $g'(x) = 5x^4$

$$\frac{d}{dx}(f(u(x))) = f'(x^5) \cdot \underbrace{u'(x)}_{5x^4} = (e^{x^5})(5x^4)$$

$$= 5x^4 \cdot e^{x^5}$$

Ex. 7.3: $f(x) = \ln(1+2x+5x^2) \Rightarrow f'(x) = ?$

$f(u) = \ln(u)$, $u(x) = 1+2x+5x^2$

$f'(u) = \frac{1}{u}$, $u'(x) = 2+10x$

$$\frac{d}{dx}\left[f(u(x))\right] = \frac{1}{u(x)} \cdot u'(x) = \left(\frac{1}{1+2x+5x^2}\right)(2+10x)$$

$$= \boxed{\frac{2+10x}{1+2x+5x^2}}$$

$$\frac{d}{dx}\left[\ln u(x)\right] = \frac{u'(x)}{u(x)}$$

Ex. 7.4: (HW)

a) $f(x) = \sqrt{2x-3}$

b) $f(x) = (x^3+e^x)^7$

c) $f(x) = \ln\left(\frac{1}{2x+1}\right)$

Ex. 7.5: $f'(x) = ?$

a) $\boxed{f(x) = e^{ax}}$

$f(u) = e^u$, $u(x) = ax$

$f'(u) = e^u$, $u'(x) = a$

$$\Rightarrow \frac{d}{dx}(e^{ax}) = e^{ax} \cdot (a) = a e^{ax}$$

In general;

$$\boxed{\frac{d}{dx}(e^{u(x)}) = e^{u(x)} \cdot u'(x)}$$

b) $\boxed{f(x) = \ln(ax)}$

$f(u) = \ln(u)$, $u(x) = ax$

$f'(u) = \frac{1}{u}$, $u'(x) = a$

$$\Rightarrow \frac{d}{dx}(\ln(ax)) = \frac{1}{\cancel{a}x} \cdot \cancel{a} = \boxed{\frac{1}{x}}$$

7.14) $f(x) = \ln(3x) \Rightarrow f'(x) = \frac{1}{3x} \cdot 3 = \boxed{\frac{1}{x}}$

$$\boxed{f''(x) = \frac{-1}{x^2}}$$

c) $\boxed{f(x)=e^{x^2-x}}$

$$\begin{cases} f(u)=e^u, & u(x)=x^2-x \\ f'(u)=e^u, & u'(x)=2x-1 \end{cases}$$

$$\Rightarrow \frac{d}{dx}\left(e^{x^2-x}\right)=e^{x^2-x}\cdot(2x-1)$$

$$=\boxed{(2x-1)e^{x^2-x}}$$

d) $\boxed{f(x)=\ln(x^8)}$

$$f'(x)=\frac{1}{x^8}\cdot\frac{d}{dx}(x^8)$$

$$=\frac{1}{\cancel{x^8}}\cdot(8\cancel{x^7})=\boxed{\frac{8}{x}}$$

or $f(x)=\ln(x^8)=8\ln x$

$$f'(x)=8\,\frac{1}{x}=\boxed{\frac{8}{x}}$$

In general, for $a>0,\ a\neq 1$:

i) $\boxed{f(x)=a^{u(x)} \Rightarrow f'(x)=?}$

$$y=a^{u(x)} \Rightarrow \ln y=\ln a^{u(x)}=(u(x))(\ln a)$$

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}(\ln a)(u(x))=(\ln a)\frac{d}{dx}(u(x))$$

$$\frac{y'}{y}=(\ln a)\cdot u'(x) \Rightarrow y'=(\underbrace{y}_{a^{u(x)}})\cdot(\ln a)\,u'(x)$$

$$\Rightarrow \boxed{\frac{d}{dx}\left(a^{u(x)}\right)=a^{u(x)}\cdot(\ln a)\cdot u'(x)}$$

ii) $\boxed{\log_a u(x)} \underset{\text{change of base}}{=} \frac{\ln u(x)}{\ln a}$

$$\boxed{\frac{d}{dx}\left[\log_a u(x)\right]}=\frac{d}{dx}\left[\frac{\ln u(x)}{\ln a}\right]=\boxed{\frac{1}{\ln a}\overbrace{\frac{d}{dx}\left[\ln u(x)\right]}^{\frac{u'(x)}{u(x)}}}$$

p54/ Find $f'(x)$:

7.4) $f(x)=(5+x+2x^3)^7 \Rightarrow \boxed{f'(x)=7(5+x+2x^3)^6\cdot(1+6x^2)}$

7.5) $f(x)=\dfrac{x}{\sqrt{3x^2+2}}=\dfrac{x}{(3x^2+2)^{1/2}}\Big\}$ quotient

$\Rightarrow$ Using quotient rule

$$f'(x)=\frac{(1)(3x^2+2)^{1/2}-\left[\left(\frac{1}{2}\right)(3x^2+2)^{\frac{1}{2}-1}(6x)\right](x)}{\left[(3x^2+2)^{1/2}\right]^2}$$

$$=\frac{\sqrt{3x^2+2}-\dfrac{\overset{3}{\cancel{6}}x^2}{\cancel{2}\sqrt{3x^2+2}}}{(3x^2+2)}=\frac{\dfrac{(3x^2+2)-3x^2}{\sqrt{3x^2+2}}}{(3x^2+2)}=\boxed{\frac{2}{(3x^2+2)^{3/2}}}$$

If $f'(x_0)=\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}$ exists

$\Rightarrow f$ is differentiable at x_0

$a>0,\ a\neq 1$

i) $\dfrac{d}{dx}\left(a^{u(x)}\right)=a^{u(x)}\ln a\ u'(x)$

ii) $\dfrac{d}{dx}\left(\log_a u(x)\right)=\dfrac{u'(x)}{u(x)}\,\dfrac{1}{\ln a}$

$$\frac{\frac{a}{b}}{c}=\frac{a}{bc}$$

7.8) $f(x) = (e^{3x}+1)^5$

$f'(x) = 5(e^{3x}+1)^4 \cdot (e^{3x} \cdot 3)$

$= \boxed{15 e^{3x} (e^{3x}+1)^4}$

NOT!!!

$\cancel{f'(x) = 5(3e^{3x}+1)^4}$

7.10) $f(x) = \sqrt{x^2+2e^{3x}}$

$f(x) = (x^2+2e^{3x})^{1/2}$

$f'(x) = \frac{1}{2}(x^2+2e^{3x})^{\frac{1}{2}-1} \cdot (2x+2e^{3x} \cdot 3)$

$= \frac{1}{\cancel{2}\sqrt{x^2+2e^{3x}}} \cdot \cancel{2}(x+3e^{3x})$

$\Rightarrow \boxed{f'(x) = \frac{x+3e^{3x}}{\sqrt{x^2+2e^{3x}}}}$

Find f'':

7.13) $f(x) = 5^{2x}$ $\quad \begin{pmatrix} a>0 \\ a \neq 1 \end{pmatrix}$

$\boxed{f(x) = a^{u(x)} \Rightarrow f'(x) = a^{u(x)} \cdot u'(x) \ln a}$

$f'(x) = 5^{2x} \cdot (2) \cdot \ln 5$

$\boxed{f'(x) = (2\ln 5)(5^{2x})}$

$f''(x) = (2\ln 5)(5^{2x} \cdot (2)(\ln 5))$

$= 4(\ln 5)^2 \cdot 5^{2x}$

$= \boxed{4(\ln^2 5) \cdot 5^{2x}}$

7.16 $f(x) = x^7 \cdot e^{-x}$

Using product rule:

$f'(x) = (7x^6)(e^{-x}) + (x^7)(e^{-x} \cdot (-1))$

$= 7x^6 e^{-x} - x^7 e^{-x}$

$f'(x) = \boxed{\underbrace{x^6}_{1} \underbrace{e^{-x}}_{2} \underbrace{[7-x]}_{3}}$

$f''(x) = (6x^5)[e^{-x} \cdot (7-x)]$

$+ (x^6)[-e^{-x}](7-x)$

$+ x^6 \cdot e^{-x} \cdot (-1)$

$f''(x) = x^5 \cdot e^{-x}[6(7-x) - x(7-x) - x]$

$= x^5 \cdot e^{-x}[\underbrace{42-6x-7x+x^2-x}_{42-14x+x^2}]$

$f''(x) = x^5 e^{-x}[42-14x+x^2] = \boxed{e^{-x}[42x^5-14x^6+x^7]}$

7.12 $f(x) = x \cdot e^x \cdot \underbrace{\log_3(x+x^4)}_{\frac{\ln(x+x^4)}{\ln 3}}$ change of base formula

$\Rightarrow f(x) = \frac{1}{\ln 3}\left\{(x)(e^x)[\ln(x+x^4)]\right\}$

$\boxed{f'(x)} = \frac{1}{\ln 3}\left\{(1)(e^x)(\ln(x+x^4)) + (x)(e^x) \cdot (\ln(x+x^4)) + (x)(e^x)\left(\frac{(1+4x^3)}{x+x^4}\right)\right\}$

$= e^x \cdot \underbrace{\frac{\ln(x+x^4)}{\ln 3}}_{\log_3(x+x^4)} + x \cdot e^x \cdot \underbrace{\frac{\ln(x+x^4)}{\ln 3}}_{\log_3(x+x^4)} + \cancel{x} \cdot e^x \cdot \underbrace{\left(\frac{1+4x^3}{x+x^4}\right)}_{\cancel{x}(1+x^3)}$

$\boxed{f'(x) = (e^x + xe^x)\log_3(x+x^4) + e^x\left(\frac{1+4x^3}{1+x^3}\right)}$