Logarithmic Differentiation:

eg. $\boxed{y=\dfrac{(x^3+1)(x^2-1)}{x^8+6x^4+1}}$; find $y'=?$

Instead of using the quotient/product rule if we transform the above function using logarithms into sums (differences) and use

$$\boxed{\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}}$$

we can find y' more easily.

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$$\boxed{\ln A^r = r \ln A}$$

$$\ln(A\cdot B) = \ln A + \ln B$$

$$\ln\left(\frac{C}{D}\right) = \ln C - \ln D$$

$$\Rightarrow \boxed{\ln y} = \ln\left(\frac{(x^3+1)(x^2-1)}{x^8+6x^4+1}\right)$$

$$= \boxed{\ln(x^3+1) + \ln(x^2-1) - \ln(x^8+6x^4+1)}$$

Differentiating both sides, we get:

$$\frac{y'}{\boxed{y}} = \left(\frac{3x^2}{x^3+1} + \frac{2x}{x^2-1} - \frac{8x^7+24x^3}{x^8+6x^4+1}\right)$$

$$\downarrow$$

$$\Rightarrow y' = y\left(\quad\right)$$

$$\Rightarrow y' = \left[\frac{(x^3+1)(x^2-1)}{x^8+6x^4+1}\right]\left[\frac{3x^2}{x^3+1} + \frac{2x}{x^2-1} - \frac{8x^7+24x^3}{x^8+6x^4+1}\right]$$

Ex. 7.6: $y = f(x) = x^x \Rightarrow y' = ?$

$$\ln y = \ln x^x = (x)(\ln x) \Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}\overbrace{(x\cdot \ln x)}^{\text{product rule}}$$

$$\Rightarrow \frac{y'}{y} = (1)(\ln x) + (x)\underbrace{\left(\frac{1}{x}\right)}_{\frac{d}{dx}(\ln x)} = (\ln x + 1)$$

$$\Rightarrow y' = y(\ln x + 1) = \boxed{(x^x)(\ln x + 1)} = (\ln x + 1)(x^x)$$

e.g. $y = a^x \quad (a>0,\ a\neq 1)$

$$\ln y = \ln a^x = x\cdot(\ln a)$$

$$\Rightarrow \frac{y'}{y} = (\ln a)\cdot\underbrace{\frac{d}{dx}(x)}_{1} = \ln a \Rightarrow y' = y \ln a$$

$$\boxed{\frac{d}{dx}(a^x) = a^x \cdot \ln a}$$

Ex. 7.7: $y = x^{\ln x} \Rightarrow y' = ?$

$$\ln y = \ln x^{\ln x} = (\ln x)(\ln x) = (\ln x)^2 \neq \underbrace{\ln x^2}_{2\ln x}$$

$$\neq \cancel{(2\ln x)}$$

$$\ln y = (\ln x)^2 \Rightarrow \frac{y'}{y} \underset{\downarrow}{=} 2(\ln x)\cdot\left(\frac{1}{x}\right)$$

using chain rule

$$\Rightarrow y' = y\left(2(\ln x)\left(\frac{1}{x}\right)\right)$$

$$\boxed{y' = (x^{\ln x})\,\frac{2\ln x}{x}}$$

$$\boxed{y' = (x^{\ln x})\,\frac{(\ln x^2)}{x}}$$

$$\Rightarrow y' = (2\ln x)(x^{\ln x - 1})$$

Ex 7.8: $y = (x+e^x)^{\ln x}$

Taking the logarithm of both sides and using the properties given in (***):

$$\boxed{\ln y} = \ln(x+e^x)^{\ln x} = \boxed{(\ln x)(\ln(x+e^x))}$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[(\ln x)(\ln(x+e^x))\right]$$

$$\frac{y'}{y} \underset{\downarrow}{=} \left(\frac{1}{x}\right)\left[\ln(x+e^x)\right] + (\ln x)\cdot\left[\frac{1+e^x}{x+e^x}\right]$$

using product rule

$$y' = y\left[\frac{\ln(x+e^x)}{x} + \frac{(\ln x)(1+e^x)}{x+e^x}\right]$$

$$y' = (x+e^x)^{\ln x}\left[\frac{\ln(x+e^x)}{x} + \frac{(\ln x)(1+e^x)}{(x+e^x)}\right]$$

Find f' using logarithmic differentiation:

p.54 7.18) $f(x) = \dfrac{(3x^4+x^2)^6}{(1+x+x^2)^8} \Rightarrow f' = ?$

$$\boxed{\ln f(x)} = \ln\left[\frac{(3x^4+x^2)^6}{(1+x+x^2)^8}\right] = \ln(3x^4+x^2)^{6} - \ln(1+x+x^2)^{8}$$

$$= \boxed{6\ln(3x^4+x^2) - 8\ln(1+x+x^2)}$$

logarithmic differentiation:

$$\frac{f'(x)}{f(x)} = 6\cdot\frac{12x^3+2x}{3x^4+x^2} - 8\,\frac{(1+2x)}{1+x+x^2}$$

$$\boxed{f'(x)} = \boxed{f(x)}\left[\frac{(72x^3+12x)}{3x^4+x^2} - \frac{(8+16x)}{1+x+x^2}\right] = \boxed{\frac{(3x^4+x^2)^6}{(1+x+x^2)^8}\cdot\left[\frac{72x^3+12x}{3x^4+x^2} - \frac{8+16x}{1+x+x^2}\right]}$$

7.19) $f(x)=(\ln x)^x \Rightarrow f'(x)=?$

$$\ln f(x)= \ln(\ln x)^{x} = (x)\cdot[\ln(\ln x)]$$

$$\Rightarrow \left(\frac{f'(x)}{f(x)}\right) = (1)\left(\ln(\ln x)\right) + (x)\,\frac{\left(\frac{1}{x}\right)}{\ln x}$$

$$= \left(\ln(\ln x) + \frac{1}{\ln x}\right)$$

$$\Rightarrow f'(x) = \underbrace{f(x)}_{(\ln x)^x}\left[\ln(\ln x) + \frac{1}{\ln x}\right]$$

$$\Rightarrow f'(x) = (\ln x)^x\left[\ln(\ln x) + \frac{1}{\ln x}\right]$$

$$f'(x) = (\ln x)^x \cdot (\ln(\ln x)) + (\ln x)^{x-1}$$

7.22) $f(x) = x^2(1-x)^2, \; x_0 = 2$

Find the eqn. of the tangent line to $f(x)$ at $x_0=2$

Point: $(x_0, f(x_0)) = (2, f(2)) = \boxed{(2, 4)}$

$$f(2) = 2^2(1-2)^2 = 4(-1)^2 = 4$$

$$f'(x) = (2x)(1-x)^2 + x^2\left(2(1-x)(-1)\right)$$

$$\underbrace{f'(2)}_{\text{slope of tg line}} = (2(2))(1-2)^2 + (2)^2\left(-2(1-2)\right) = 4 + 8 = \boxed{12}$$

tg. line: $y - 4 = (12)(x-2) = 12x - 24$

$$y = 12x - 24 + 4 \Rightarrow \boxed{y = 12x - 20}$$

7.29) $f(x) = \sqrt{10 - e^{-x}}, \; a = 0 \Rightarrow f'(0) = ?$

$$f(x) = (10 - e^{-x})^{1/2} \Rightarrow f'(x) = \frac{1}{2}(10 - e^{-x})^{-1/2} \cdot \left(-e^{-x}(-1)\right)$$

$$\Rightarrow f'(x) = \frac{e^{-x}}{2\sqrt{10 - e^{-x}}} \Rightarrow f'(0) = \frac{e^{-0}}{2\sqrt{10 - e^{-0}}} = \frac{1}{2\underbrace{\sqrt{10-1}}_{\sqrt{9}=3}} = \frac{1}{2\cdot 3} = \boxed{\frac{1}{6}}$$

7.33) $f(x) = \dfrac{1}{2 + 4x + 8e^{2x}}, \; a = 0$

$$f(x) = \frac{1}{u(x)}$$

$$f'(x) = \frac{-u'(x)}{(u(x))^2}$$

$$f'(x) = -\frac{4 + 16e^{2x}}{(2 + 4x + 8e^{2x})^2}$$

$$f'(0) = -\frac{4 + 16e^{2(0)}}{(2 + 4(0) + 8e^{2(0)})^2} = \frac{-4+16}{(10)^2} = \frac{-20}{100} = \frac{-1}{5}$$

7.34) $f(x) = x \ln\sqrt{1+2x}$, $a=1$

$$f(x) = (x)\ln(1+2x)^{1/2} = \left[\frac{x}{2}\right]\left[\ln(1+2x)\right]$$

$$= \left(\frac{1}{2}\right)\left[x \cdot \ln(1+2x)\right]$$

$$f'(x) = \frac{1}{2}\left[(1)\cdot\ln(1+2x) + (x)\cdot\frac{2}{1+2x}\right]$$

$$f'(x) = \frac{\left[\ln(1+2x) + \frac{2x}{1+2x}\right]}{2}$$

$$f'(1) = \frac{\ln 3 + \frac{2}{3}}{2} = \boxed{\frac{3\ln 3 + 2}{6}}$$

$$= \frac{\ln 27 + 2}{6}$$

7.35) $f(x) = \ln\left(\frac{xe^x}{1+x^2}\right)$, $a=2 \Rightarrow f'(2) = ?$

$$f(x) = \ln(xe^x) - \ln(1+x^2)$$

$$= \ln x + \underbrace{\ln e^x}_{x} - \ln(1+x^2)$$

$$f(x) = \ln x + x - \ln(1+x^2)$$

$$f'(x) = \frac{1}{x} + 1 - \frac{2x}{1+x^2}$$

$$f'(2) = \frac{1}{2} + 1 - \frac{4}{5}$$

$$= \frac{3}{2} - \frac{4}{5} = \frac{15-8}{10} = \frac{7}{10}$$

Find $y'(x) = ?$ if $y(x) = \sqrt[3]{\frac{6(x^3-1)^2}{x^6 e^{-4x}}} = \left(\frac{6(x^3-1)^2}{x^6 e^{-4x}}\right)^{1/3}$

$$\ln y = \frac{1}{3}\ln\left[\frac{6(x^3-1)^2}{x^6 e^{-4x}}\right] = \frac{1}{3}\left[\ln 6 + 2\ln(x^3-1) - 6\ln x - \overbrace{\underbrace{\ln e^{-4x}}_{-4x}}^{4x}\right]$$

diff:

$$\frac{y'}{y} = \frac{1}{3}\left[0 + 2\frac{3x^2}{x^3-1} - \frac{6}{x} + 4\right] = \frac{1}{3}\left[\underset{(x)}{\frac{6x^2}{x^3-1}} - \underset{(x^3-1)}{\frac{6}{x}} + \underset{x(x^3-1)}{4}\right]$$

$$y' = y\left[\frac{6x^3 - 6(x^3-1) + 4x(x^3-1)}{3x(x^3-1)}\right] = y\left[\frac{4x^4 - 4x + 6}{3x(x^3-1)}\right]$$

$$\boxed{y' = \sqrt[3]{\frac{6(x^3-1)^2}{x^6 \cdot e^{-4x}}}\left(\frac{4x^4 - 4x + 6}{3x(x^3-1)}\right)}$$