Inverse Functions:

If $\boxed{f(g(x))=x}$, $\boxed{g(f(x))=x} \Rightarrow$
f and g are said to be inverses of each other.

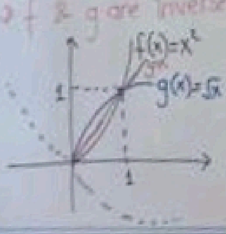
eg. $\boxed{f(x)=x^2,\ g(x)=\sqrt{x}}$

$f(g(x))=f(\sqrt{x})=(\sqrt{x})^2=\boxed{x}$

$g(f(x))=g(x^2)=\sqrt{x^2}=|x|=\boxed{x}$

since dom. of g is R_+ $[0,\infty)$

$\Rightarrow$ f & g are inverses of each other



$y=x^2 \longleftrightarrow x=y^2 \longrightarrow \boxed{y=\sqrt{x}}$

interchange x & y — solve for y — resulting function is the inverse of the starting func., denoted by: $\boxed{f^{-1}(x)}$

$f(x)=x^2 \longrightarrow \underbrace{f^{-1}}_{\text{inverse func. of } f}(x)=\sqrt{x}$

p.25 Find the inverse of the following functions:

3-12) $\boxed{f(x)=\dfrac{x+2}{5x+4}}$

$\text{Dom} f: \{x \mid x \neq -\tfrac{4}{5}\} = (-\infty,-\tfrac{4}{5}) \cup (-\tfrac{4}{5},\infty)$

$y=\dfrac{x+2}{5x+4} \xleftrightarrow{x \leftrightarrow y} x=\dfrac{y+2}{5y+4} \longrightarrow x(5y+4)=y+2$

$y(5x-1)=2-4x$

$y=\dfrac{2-4x}{5x-1} \Rightarrow \boxed{f^{-1}(x)=\dfrac{2-4x}{5x-1}}$ ✓

*Note: the domain and range of the original function and its inverse are reversed.

Check that: i) $f(f^{-1}(x))=x$? ✓

ii) $f^{-1}(f(x))=x$?

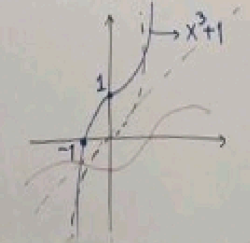
i) $\boxed{f(f^{-1}(x))}=f\left(\dfrac{2-4x}{5x-1}\right)=\dfrac{\left(\dfrac{2-4x}{5x-1}\right)+2}{5\left(\dfrac{2-4x}{5x-1}\right)+4}=\dfrac{\dfrac{\cancel{2}-4x+2(5x-\cancel{1})}{5x-1}}{\dfrac{10-\cancel{20x}+\cancel{20x}-4}{5x-1}}$

$=\dfrac{6x}{\cancel{5x-1}}\cdot\dfrac{\cancel{5x-1}}{6}=\boxed{x}$ ✓

ii) $\boxed{f^{-1}(f(x))} = f^{-1}\left(\frac{x+2}{5x+4}\right) = \dfrac{2-4\left(\frac{x+2}{5x+4}\right)}{5\left(\frac{x+2}{5x+4}\right)-1}$

$$= \dfrac{\frac{10x+8-4x-8}{5x+4}}{\frac{5x+10-5x-4}{5x+4}} = \frac{6x}{\cancel{5x+4}} \cdot \frac{\cancel{5x+4}}{6} = \boxed{x} \checkmark$$

3-14) $\boxed{f(x) = x^{3}+1}$



$y = x^3+1 \xleftrightarrow[x \leftrightarrow y]{} x = y^3+1 \rightarrow y^3 = x-1$

$\rightarrow \boxed{y = \sqrt[3]{x-1}}\ f^{-1}(x)$

Claim: i) $f(f^{-1}(x)) = x$? ✓

ii) $f^{-1}(f(x)) = x$? ✓

i) $f(f^{-1}(x)) = f(\sqrt[3]{x-1}) = (\sqrt[3]{x-1})^3+1 = (x-1)+1 = \boxed{x}$ ✓

ii) $f^{-1}(f(x)) = f^{-1}(x^3+1) = \sqrt[3]{(x^3+1)-1} = \sqrt[3]{x^3} = \boxed{x}$ ✓

$f(x) = x^3+1$ & $f^{-1}(x) = \sqrt[3]{x-1}$

are inverses of each other.

One-to-one functions:

If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \Rightarrow f$ is 1-1

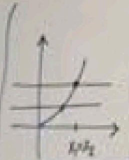
eg $f(x) = x^2$: $\underbrace{f(x_1)}_{x_1^2} = \underbrace{f(x_2)}_{x_2^2} \Rightarrow x_1^2 = x_2^2$

$x_1^2 - x_2^2 = 0$

$\left.\begin{array}{l} x_1 > 0 \\ x_2 > 0 \end{array}\right\}$ Dom: $[0,\infty)$

$\underbrace{(x_1-x_2)}_{=0}\underbrace{(x_1+x_2)}_{\neq 0} = 0$

$\Rightarrow x_1 = x_2$



$f(x) = x^2$ is not 1-1 on $(-\infty,\infty)$ [but it is on $[0,\infty)$]

$g(x) = x^3$ is 1-1 on $(-\infty,\infty)$

Onto function: Let $f: A \to B$. If $\exists$ an $x \in A$ $\forall y \in B$ st $f(x) = y \Rightarrow$ f is onto

ex. $f(x) = 2x+1$ onto ✓

$g(x) = |x|$ not onto (since $\nexists x \in \mathbb{R} \ni g(x) = -2$).

Thm: A function has an inverse $\Leftrightarrow$ if it is 1-1 and onto

Ex 3-3 Find the inverse of the function $f(x) = \frac{x-2}{x+1}$ on domain: $\mathbb{R} \setminus \{-1\}$ $\left((-\infty,-1) \cup (-1,\infty)\right)$

and $\boxed{\text{range } \mathbb{R} \setminus \{1\}}$ (this function is 1-1 & onto in the given domain & range)

Soln: $y = \frac{x-2}{x+1} \xleftrightarrow[x \leftrightarrow y]{} x = \frac{y-2}{y+1} \rightarrow xy + x = y-2$

$y(x-1) = -2-x$

$\boxed{y = -\dfrac{2+x}{x-1}} = f^{-1}(x)$

Domain $f^{-1}(x)$: $\mathbb{R} \setminus \{1\}$

Range $\mathbb{R} \setminus \{-1\}$

HW: Check that:

i) $f(f^{-1}(x)) = x$

ii) $f^{-1}(f(x)) = x$