Midterm 1: Nov. 17, 2022; 17:30 (Thursday)

Midterm 2: Dec. 22, 2022; 17:30 (Thursday)

Exponential Functions:

Functions of the form

$$\boxed{f(x) = a^x} \quad \begin{pmatrix} a>0 \\ a\neq 1 \end{pmatrix}$$

$(5)^{-8} = \frac{1}{5^8} > 0$

are called exponential functions.

Domain of $f(x)=a^x$ is $(-\infty,\infty)$, (or $\mathbb{R}$);

Range of $f(x)=a^x \begin{pmatrix} a>0 \\ a\neq 1 \end{pmatrix}$ is $(0,\infty)$

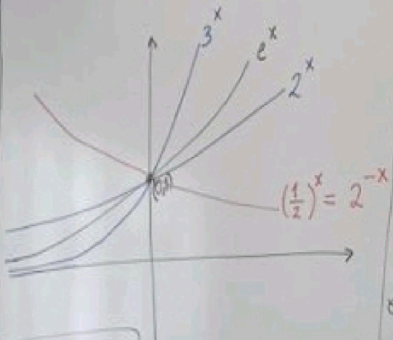
Remember the algebraic properties:

- $a^n = \underbrace{a \cdot \ldots \cdot a}_{n\text{-times}}$
- $a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$
- $a^{1/n} = \sqrt[n]{a}$
- $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Consider the natural base number

$e \cong 2.718 \ldots$

$\Rightarrow \boxed{f(x)=e^x} \quad \begin{pmatrix} e>0 \\ e\neq 1 \end{pmatrix}$ natural exponential function



$\lim_{x\to-\infty} (a^x) = 0$ $\begin{pmatrix} a>0 \\ a\neq 1 \end{pmatrix}$

$3 > e > 2 \Rightarrow \frac{1}{3} < \frac{1}{e} < \frac{1}{2}$

*Note; eg. $f(x)=2^x$, $g(x)=x^2$

Do not confuse the exponential function $f(x)$ with the polynomial function $g(x)$.

Logarithmic Functions:

Observing that $f(x)=a^x$ $(a>0, a\neq 1)$ is 1-1 & onto $\Rightarrow$ we can speak about its inverse function $\Rightarrow$ namely the logarithmic function; $\Rightarrow$ $\boxed{f(x)=\log_a x}$, $a>0, a\neq 1$

$\Rightarrow \boxed{a^{\log_a x} = \log_a a^x = x}$

Special logarithmic functions:

$a=10 \Rightarrow \log_{10} x = \log x$ (common logarithm)

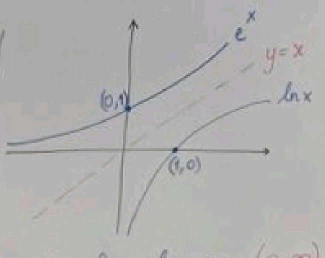
$a=e \Rightarrow \log_e x = \ln x$ (natural logarithm)

Since; $a^x \cdot a^y = a^{x+y}$

$\Rightarrow \boxed{\log_a (A \cdot B) = \log_a A + \log_a B}$

This property leads to: $\left(\begin{array}{c} a>0,\ a\neq 1 \\ B\neq 0 \end{array}\right)$

* $\log_a \left(\frac{A}{B}\right) = \log_a A - \log_a B$
* $\log_a (A^r) = r(\log_a A)$
* $\log_a \left(\frac{1}{B}\right) = -\log_a B$



Domain of $y=\ln x$ is: $(0,\infty)$

Range of " ": $\mathbb{R}$

Change of base formula:

$\left(\begin{array}{cc} a>0, & b>0 \\ a\neq 1 & b\neq 1 \end{array}\right)$

$$\log_a x = \frac{\log_b x}{\log_b a} \underset{b=e}{=} \frac{\ln x}{\ln a}$$

* Any exponential can be expressed in terms of the natural exp:

$\boxed{a^x = e^{x\ln a}}\ \left(= e^{\ln a^x} = a^x\right)$

Ex 3-6: Simplify $\log 360$.

360	2
180	2
90	2
45	5
9	3
3	3
1	1

$360 = 2^3 \cdot 3^2 \cdot 5$

$\log(360) = \log(2^3 \cdot 3^2 \cdot 5)$

$= \log 2^3 + \log 3^2 + \log 5$

$= 3\log 2 + 2\log 3 + \underbrace{\log 5}_{1-\log 2}$

$5 = \frac{10}{2}$

$\log 5 = \underbrace{\log 10}_{1} - \log 2$

$\boxed{\log_a a = 1 \quad (a>0,\ a\neq 1)}$

$= 1 + (3-1)\log 2 + 2\log 3$

$= \boxed{1 + 2\log 2 + 2\log 3}$

ify the following:

$$\log_8 16 = \frac{\log_2 16}{\log_2 8} = \frac{\log_2 2^4}{\log_2 2^3} = \frac{4 \cdot \overbrace{\log_2 2}^{1}}{3 \cdot \underbrace{\log_2 2}_{1}} = \boxed{\frac{4}{3}}$$

change the base to 2

$\log_2 2 = 1$

$$\log_3\left(\frac{\sqrt{3}}{81}\right) = \log_3 (3)^{1/2} - \log_3 (3)^4$$

$\log_3 3 = 1$

$$\left(\tfrac{1}{2}\right)\underbrace{\log_3 3}_{1} - (4)\underbrace{\log_3 3}_{1} = \frac{1}{2} - 4 = \boxed{-\frac{7}{2}}$$

$$e^{2x+5\ln x} = e^{2x} \cdot e^{5 \ln x} = e^{2x} \cdot \underbrace{e^{\ln(x^5)}}_{x^5}$$

$$= \boxed{x^5 \cdot e^{2x}}$$

3-24) $3^{2\log_9 x} = 3^{\log_9 x^2} = 3^{\log_3 x} = \boxed{x}$

method 1

$$\left(\log_9 x^2 = \frac{\log_3 x^2}{\underbrace{\log_3 9}_{2 \log_3 3} = 2} = \frac{1}{2}\log_3 x^2 = \log_3 \sqrt{x^2} = \log_3 x \right)$$

Method 2: $(a^m)^n = a^{m \cdot n}$

$$3^{2\log_9 x} = (3^2)^{\log_9 x}$$

$$= (9)^{\log_9 x} = \boxed{x}$$

$\left(a^{\log_a x}\right) = x$

$(a > 0, a \neq 1)$

3-25) $5^{\log_{25} x} = 5^{\log_5 \sqrt{x}} = \boxed{\sqrt{x}}$

$$\log_{25} x = \frac{\log_5 x}{\log_5 (25) = 2} = \frac{1}{2}\log_5 x = \log_5 \sqrt{x}$$

$2(\underbrace{\log_5 5}_{1})$

3-26)

$$10^{1+\log(2x)} = 10^1 \cdot \underbrace{10^{\log(2x)}}_{\underbrace{10^{\log_{10}(2x)}}_{2x}}$$

$$= (10)(2x) = \boxed{20x}$$

Solve the following equations:

3-28) $\log_x 12 = \left(\frac{1}{2}\right)$ $\left(\begin{matrix} x > 0 \\ x \neq 1 \end{matrix}\right)$

$$\Leftrightarrow \left(x^{1/2}\right)^2 = (12)^2 \Rightarrow \boxed{x = 144}$$

3-31) $\log_x 64 = 4$

$$\Leftrightarrow x^4 = 64$$

$$x = \sqrt[4]{64} = \left[(2\sqrt{2})^4\right]^{1/4}$$

$$\boxed{x = 2\sqrt{2}}$$

eg: $\log_x 1 = 2$ $(x > 0, x \neq 1)$

$$\Rightarrow x^2 = 1 \Rightarrow x = \mp 1$$

$\cancel{x = 1}$ ✗ $(x \neq 1)$

$\cancel{x = -1}$ ✗ $(x > 0)$

$\Rightarrow$ No soln. $(\{\phi\}$ soln.$)$

eg. $\log_x(6-5x) = 2 \iff \boxed{x^2 = 6-5x}$

$(x>0, x\neq 1)$

$x^2+5x-6=0$

$(x+6)(x-1)=0$

$\Rightarrow \cancel{x=-6},\ \cancel{x=1}$

$(x>0)\quad (x\neq 1)$

$\Rightarrow$ No soln.

(or Soln.: $\{\emptyset\}$)

3-33.) $\log_9(18x) = 2$

$\iff 9^2 = 18x \Rightarrow \frac{\cancel{81}^{9}}{\cancel{18}_{2}} = x$

$\Rightarrow \boxed{x = \frac{9}{2}}$

3-35) $\log_{10}(\log_{10}x) = 0$

$\Rightarrow \underbrace{10^0}_{1} = \log_{10}x$

$\Rightarrow \log_{10}x = 1$

$\Rightarrow x = 10^1 = 10$

$\boxed{x = 10}$

3-36.) $\ln_e(\ln x) = 1$

$e^1 = \ln x \Rightarrow \ln_e x = e$

$\Rightarrow e^e = x$

3-38) $2^{4x+4} = 8^{x-1} = (2^3)^{x-1} = 2^{3x-3}$

$\Rightarrow 4x+4 = 3x-3$

$4x-3x = -3-4$

$\boxed{x=-7}$

Simplify: $\to (x>0)$ (domain of log func. $(0,\infty)$)

$3^{(2\log_3 x)} \cdot 9^{(\log_4 2)} = 6$

$* \ 3^{2\log_3 x} = 3^{\log_3 x^2} = \boxed{x^2}$

$* \ \log_4 2 = \frac{\log_2 2}{\log_2 4} = \frac{1}{2\log_2 2} = \frac{1}{2}$

$\Rightarrow x^2 \cdot \underbrace{9^{\frac{1}{2}}}_{3} = 6$

$3x^2 = 6$

$x^2 = 2$

$\Rightarrow \boxed{x = \mp\sqrt{2}}$

$x \neq -\sqrt{2}$

$\boxed{\text{Soln.: } \{\sqrt{2}\}}$

$\log_a x = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln b}$

$\log_a a = 1 \quad (a>0, a\neq 1)$

$(a^1 = a)$

$\log_a x = b \iff a^b = x$