

Ch 12: Integrals

Antiderivative:

If f is the derivative of F , then F is the anti-derivative of f and

$$F'(x) = f(x)$$

$$\Rightarrow \int f(x) dx = F(x) + C$$

integrate integral *integrand function* *integration variable* *integration constant* *C → constant*

ex. $f(x) = x$, $F_1(x) = \frac{x^2}{2}$, $F_2(x) = \frac{x^2}{2} + 7$

$$F_3(x) = \frac{x^2}{2} - 25, \dots$$

Some basic integrals:

$$\int 1 dx = x + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int F'(x) dx = F(x) + C$$

Ex 12-1: Evaluate the integral

$$\begin{aligned} & \int \left(\frac{1}{x^3} - 2x + 4 \right) dx \\ &= \int x^{-3} dx - 2 \int x dx + 4 \int 1 dx \\ &= \frac{x^{-2}}{-2} - 2 \frac{x^2}{2} + 4x + C \\ &= \boxed{\frac{-1}{2x^2} - x^2 + 4x + C} \end{aligned}$$

Ex 12-2, 12-3: HW

Ex 12-4: Find a function

$f(x)$ st. $f''(x) = 4 - \frac{8}{x^2}$
and $f(1) = -15$, $f'(1) = 7$

Soln:

$$\begin{aligned} f'(x) &= \int f''(x) dx \\ &= \int \left(4 - \frac{8}{x^2} \right) dx = \int (4 - 8x^{-2}) dx \\ &= 4x - 8 \frac{x^{-1}}{-1} + C_1 \end{aligned}$$

$$\begin{aligned} \Rightarrow f'(x) &= 4x + \frac{8}{x} + C_1 \\ f'(1) &= 4(1) + \frac{8}{1} + C_1 \\ 7 &= 12 + C_1 \Rightarrow \boxed{C_1 = -5} \end{aligned}$$

$$f'(x) = 4x + \frac{8}{x} - 5$$

$$f(x) = \int f'(x) dx = \int (4x + \frac{8}{x} - 5) dx$$

$$f(x) = 4 \frac{x^2}{2} + 8 \ln|x| - 5x + C_2$$

$$f(1) = 2(1)^2 + 8 \ln|1| - 5(1) + C_2$$

$$-15 - 2 + 5 = C_2 \Rightarrow C_2 = -12$$

$$\Rightarrow f(x) = 2x^2 + 8 \ln|x| - 5x - 12$$

Definite Integral:

$$\int_a^b f(x) dx$$

f cont. on $[a, b]$

→ definite integral of f on $[a, b]$ corresponds to the area under $f(x)$ on $[a, b]$.

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

The Fundamental Thm. of Calculus

Ex 12-5: Evaluate $\int_0^2 (3x^2 + 8x - 5) dx = ?$

$$\int_0^2 (3x^2 + 8x - 5) dx = \left[3 \frac{x^3}{3} + 8 \frac{x^2}{2} - 5x \right]_0^2$$

$$= [(2)^3 + 4(2)^2 - 5(2)] - [(0)^3 + 4(0)^2 - 5(0)]$$

$$= 8 + 16 - 10 = 14$$

Ex 12-6: $\int_1^9 \frac{5}{\sqrt{x}} dx = 5 \int_1^9 x^{-1/2} dx$

$$= 5 \left[\frac{x^{1/2}}{1/2} \right]_1^9 = 10 [\sqrt{9} - \sqrt{1}] = 10(3-1) = 20$$

Ex 12-7: $\int_3^7 \frac{dx}{x} = \ln|x| \Big|_3^7 = \ln 7 - \ln 3 = \ln \frac{7}{3}$

P.99 Evaluate the following indefinite integrals:

8) $\int \frac{1}{\sqrt[3]{x^5}} dx = \int x^{-5/3} dx = \frac{x^{-5/3+1}}{-5/3+1} + C$

$$= \frac{x^{-2/3}}{-2/3} + C = -\frac{3}{2\sqrt[3]{x^2}} + C$$

9) $\int \frac{(2u^2 - 3u + 12)}{u^2} du = \int \left(\frac{2u^2}{u^2} - \frac{3u}{u^2} + \frac{12}{u^2} \right) du$

$$= 2 \int du - 3 \int \frac{1}{u} du + 12 \int u^{-2} du$$

$$= 2u - 3 \ln|u| + \frac{12u^{-1}}{-1} + C$$

$$= 2u - 3 \ln|u| - \frac{12}{u} + C$$

Evaluate the following definite integrals:

$$12-16) \int_{-1}^2 (1+e^x) dx = (x+4e^x) \Big|_{-1}^2$$

$$= (2+4e^2) - (-1+4e^{-1})$$

$$= 2+4e^2+1-\frac{4}{e}$$

$$= \boxed{3+4e^2-\frac{4}{e}}$$

$$12-17) \int_1^4 5t^{-2} dt = ?$$

$$= 5 \left(\frac{t^{-1}}{-1} \right) \Big|_1^4 = -5 \left[\frac{1}{4} - \frac{1}{1} \right]$$

$$= -5 \left(\frac{-3}{4} \right) = \boxed{\frac{15}{4}}$$

$$12-18) \int_1^9 \left(\frac{1-\sqrt{x}}{\sqrt{x}} \right) dx = \int_1^9 (x^{-\frac{1}{2}} - 1) dx$$

$$= \left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - x \right) \Big|_1^9 = (2\sqrt{9}-9) - (2\sqrt{1}-1)$$

$$= (6-9) - 1 = \boxed{-4}$$

$$12-20) \int_{-2}^{-1} \frac{1}{x^3} dx = \int_{-2}^{-1} x^{-3} dx$$

$$= \frac{x^{-2}}{-2} \Big|_{-2}^{-1} = -\frac{1}{2} \left[\frac{1}{(-1)^2} - \frac{1}{(-2)^2} \right]$$

$$= -\frac{1}{2} \left[\frac{1}{x^2} \right]_{-2}^{-1} = -\frac{1}{2} \left[1 - \frac{1}{4} \right] = \boxed{-\frac{3}{8}}$$

Ch 13 Substitution:

$$\text{Chain rule: } \frac{d}{dx} F(u(x)) = \frac{dF(u)}{du} \frac{du(x)}{dx}$$

$$\frac{dF}{dx} \begin{matrix} \downarrow \\ F \\ \downarrow \\ u \\ \downarrow \\ x \end{matrix} \Rightarrow \int f(u(x)) \cdot u'(x) dx = F(u(x)) + C$$

where F is the anti-derivative of f(x).

That is;

$$f'(x) = F(x).$$

ex: $\int \frac{2x dx}{x^2+1} = \int \frac{du}{u} = \ln|u| + C$

$u = x^2+1$
 $du = 2x dx$
 substitution

start with the variable "x"
 end with the variable "x"

$$= \ln(x^2+1) + C$$

Ex 13-1: Evaluate the integral:

$$\int (x^4+1)^2 \cdot 4x^3 dx$$

$u = x^4+1$
 $du = 4x^3 dx$

$$= \int u^2 \cdot du = \frac{u^3}{3} + C$$

$$= \frac{(x^4+1)^3}{3} + C$$

$u = x^4+1$

Ex 13-2: $\int e^{3x^2} \cdot x dx = ?$

$u = 3x^2$
 $du = 6x dx$
 $\frac{1}{6} du = x dx$
 $\int e^{3x^2} \cdot x dx = \int e^u \cdot \frac{1}{6} du$
 $= \frac{1}{6} \int e^u du = \frac{1}{6} e^u + C$
 $= \frac{1}{6} e^{3x^2} + C$

Ex 13-3: Evaluate the integral:

$\int (x^3+6x^2)^7 (x^2+4x) dx = \int u^7 \cdot \frac{1}{3} du$
 $u = x^3+6x^2$
 $du = (3x^2+12x) dx = 3(x^2+4x) dx$
 $\frac{1}{3} du = (x^2+4x) dx$
 $= \frac{1}{3} \int u^7 du = \frac{1}{3} \cdot \frac{u^8}{8} + C = \frac{1}{24} (x^3+6x^2)^8 + C$

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 13.6 $\int \frac{3y}{(y^2-2)^4} dy = 3 \int \frac{\frac{1}{2} du}{u^4}$

$u = y^2 - 2$
 $du = 2y dy$
 $\frac{1}{2} du = y dy$
 $= \frac{3}{2} \int u^{-4} du = \frac{3}{2} \left(\frac{u^{-3}}{-3} \right) + C = \frac{1}{2} u^{-3} + C = \frac{1}{2(y^2-2)^3} + C$

13-7 $\int \frac{2x^2+3x}{x^4+3x^2+1} dx = \int \frac{\frac{1}{2} du}{u} = \frac{1}{2} \int \frac{du}{u}$
 $u = x^4+3x^2+1$
 $du = (4x^3+6x) dx = 2(2x^3+3x) dx$
 $\frac{1}{2} du = (2x^3+3x) dx$
 $= \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|x^4+3x^2+1| + C$

13-9 $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int e^u \cdot 2 du = 2e^u + C$
 $u = \sqrt{x}$
 $du = \frac{1}{2\sqrt{x}} dx$
 $2 du = \frac{1}{\sqrt{x}} dx$
 $= 2e^{\sqrt{x}} + C$

13-16 $\int \frac{e^{\frac{1}{x}}}{x^2} dx = \int e^u (-du) = -\int e^u du$
 $-du = -\frac{1}{x^2} dx$
 $u = \frac{1}{x}$
 $= -e^u + C = -e^{\frac{1}{x}} + C$

13-17 $\int \frac{(e^x+3x^2)}{e^x+x^2+1} dx = ?$
 $u = e^x+x^2+1$
 $du = (e^x+2x) dx$
 $= \int \frac{du}{u} = \ln|u| + C = \ln|e^x+x^2+1| + C$