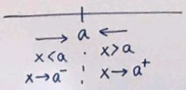
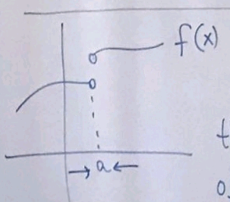
One-sided Limits:



* When x approaches "a" from values larger than "a" we use the notation $x \to a^+$

* When x approaches "a" from values less than "a" we use the notation $x \to a^-$



* If $\lim\limits_{\substack{x \to a^- \\ (x<a)}} f(x) = L_1$, then we call L_1 as the left-limit of f as $x \to a^-$ (left-sided-limit)

* If $\lim\limits_{\substack{x \to a^+ \\ (x>a)}} f(x) = L_2 \Rightarrow$ we call L_2 the right limit of f at $x = a$

Theorem:
The limit $\lim\limits_{x \to a} f(x) = L \iff$ both one-sided limits

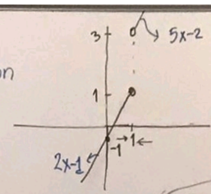
$$\lim_{x \to a^-} f(x) \quad \text{and} \quad \lim_{x \to a^+} f(x)$$

exist and are equal.

(one sided limits are finite)

Ex. 5-1: Consider the function

$$f(x) = \begin{cases} 2x-1, & \text{if } x<1 \\ 5x-2, & \text{if } x>1 \end{cases}$$



Find the limits: $\lim\limits_{\substack{x \to 1^- \\ (x<1)}} f(x) = ?$, $\lim\limits_{\substack{x \to 1^+ \\ (x>1)}} f(x) = ?$, $\lim\limits_{x \to 1} f(x) = ?$

* $\lim\limits_{\substack{x \to 1^- \\ (x<1)}} f(x) = \lim\limits_{x \to 1^-} (2x-1) = 2(1) - 1 = \boxed{1}$ left-limit of f at $x=1$

* $\lim\limits_{\substack{x \to 1^+ \\ (x>1)}} f(x) = \lim\limits_{x \to 1^+} (5x-2) = 5(1) - 2 = \boxed{3}$ right-limit of f at $x=1$.

* $\boxed{\lim\limits_{x \to 1} f(x) = \text{d.n.e.}}$ since right & left limits are not equal at $x = 1$

Ex. 5-2: Find the limits

$\lim\limits_{\substack{x\to 3^+ \\ (x>3)}} f(x) = ?$ $\lim\limits_{\substack{x\to 3^- \\ (x<3)}} f(x) = ?$ for

$f(x) = \dfrac{4x-12}{|x-3|}$ and sketch the graph of $f(x)$.

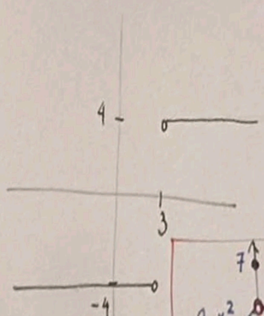
$$|x-3| = \begin{cases} x-3, & x>3 \\ -(x-3), & x<3 \end{cases}$$

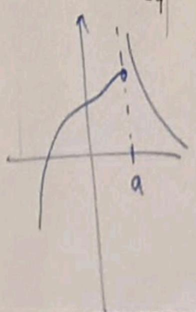
$$f(x) = \begin{cases} \dfrac{4(x-3)}{(x-3)} = \boxed{4} & x>3 \\ \dfrac{4(x-3)}{-(x-3)} = \boxed{-4} & x<3 \end{cases}$$

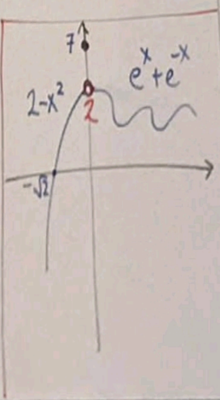
$\lim\limits_{\substack{x\to 3^- }} f(x) = \lim\limits_{\substack{x\to 3^- \\ (x<3)}} (-4) = \boxed{-4}$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{\substack{x\to 3^+ \\ (x>3)}} (4) = \boxed{4}$

$\neq \Rightarrow \boxed{\lim\limits_{x\to 3} f(x) = \text{d.n.e}}$







Ex. 5-3: $f(x) = \begin{cases} 2-x^2 & \text{if } x<0 \\ 7 & \text{if } x=0 \\ e^x+e^{-x} & \text{if } x>0 \end{cases}$

$\lim\limits_{x\to 0^-} f(x) = ?$, $\lim\limits_{x\to 0^+} f(x) = ?$, $\lim\limits_{x\to 0} f(x) = ?$

$* \lim\limits_{\substack{x\to 0^- \\ (x<0)}} f(x) = \lim\limits_{x\to 0^-} (2-x^2) = \boxed{2} \rightarrow$ left-limit of f at $x=0$.

$* \lim\limits_{\substack{x\to 0^+ \\ (x>0)}} f(x) = \lim\limits_{x\to 0^+} (e^x+e^{-x}) = e^0+e^{-0} = 1+1 = \boxed{2} \rightarrow$ right-limit of f at $x=0$

$\Rightarrow \boxed{\lim\limits_{x\to 0} f(x) = 2}$ (since right & left limits are both equal to 2 at $x=0$)