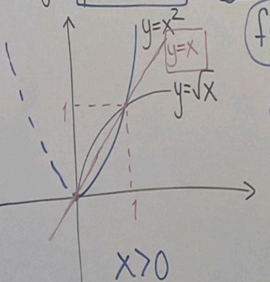
Inverse Functions:

If $f(g(x))=x$ and $g(f(x))=x \Rightarrow$ f and g are said to be inverses of each other.

The inverse of $f(x)$ is denoted by $f^{-1}(x)$.

e.g. $\boxed{f(x)=x^2}$, $\boxed{g(x)=\sqrt{x}}$



$x>0$

$\boxed{f(g(x))=f(\sqrt{x})=(\sqrt{x})^2=x}$

$g(f(x))=g(x^2)=\sqrt{x^2}=|x|=x$

$x>0$

Find the inverse of the following functions:

3-12) $\boxed{f(x)=\frac{x+2}{5x+4}}$ $D_f=\{x \mid x\neq -\frac{4}{5}\}=\mathbb{R}\setminus\{-\frac{4}{5}\}$

$y=\frac{x+2}{5x+4} \xrightarrow[\text{(interchange)}]{x\leftrightarrow y} x=\frac{y+2}{5y+4} \longrightarrow 5xy+4x=y+2$

solve for $y=f^{-1}(x)$

$\Rightarrow y(5x-1)=2-4x \Rightarrow \boxed{y=\frac{2-4x}{5x-1}}$ $f^{-1}(x)$ ← inverse function.

Check: i) $f(f^{-1}(x))=x$? ✓

ii) $f^{-1}(f(x))=x$? ✓

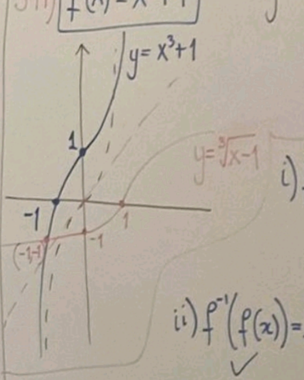
i) $\boxed{f(f^{-1}(x))}=f\left(\frac{2-4x}{5x-1}\right)=\frac{\frac{2-4x}{5x-1}+2}{5\left(\frac{2-4x}{5x-1}\right)+4}$

$=\frac{\frac{\cancel{2}-4x+10x-\cancel{2}}{5x-1}}{\frac{10-\cancel{20x}+\cancel{20x}-4}{5x-1}}=\frac{\cancel{6}x}{\cancel{5x-1}}\cdot\frac{\cancel{5x-1}}{\cancel{6}}=\boxed{x}$

ii) $\boxed{f^{-1}(f(x))}=f^{-1}\left(\frac{x+2}{5x+4}\right)=\frac{2-4\left(\frac{x+2}{5x+4}\right)}{5\left(\frac{x+2}{5x+4}\right)-1}=\frac{2(5x+4)-4(x+2)}{\cancel{5x+4}}\cdot\frac{\cancel{5x+4}}{\cancel{5x}+10-\cancel{5x}-4}$

$=\frac{10x-4x}{6}=\frac{\cancel{6}x}{\cancel{6}}=\boxed{x}$

3-14) $\boxed{f(x)=x^3+1}$



$y=x^3+1 \xleftrightarrow[x\leftrightarrow y]{} x=y^3+1 \Rightarrow y^3=x-1$

$\boxed{y=\sqrt[3]{x-1}}$ $f^{-1}(x)$

✓

i) $f(f^{-1}(x))=f(\sqrt[3]{x-1})$
$=(\sqrt[3]{x-1})^3+1=x-1+1=\boxed{x}$ ✓

ii) $f^{-1}(f(x))=f^{-1}(x^3+1)=\sqrt[3]{(x^3+1)-1}=\sqrt[3]{x^3}=\boxed{x}$ ✓

✓

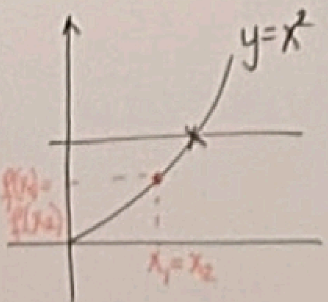
$\Rightarrow f^{-1}(x)=\sqrt[3]{x-1}$ is the inverse of $f(x)=x^3+1$.

Domain & Range of both functions are $(-\infty,\infty)$.

One-to-one Function:

If $f(x_1)=f(x_2) \Rightarrow x_1=x_2 \Rightarrow f$ is 1-1

$y=x^2$ on $[0,\infty)$



$f(x_1)=f(x_2) \Rightarrow x_1^2=x_2^2 \Rightarrow$

$x_1^2-x_2^2=0 \Rightarrow$

$\underbrace{(x_1-x_2)}_{=0}\underbrace{(x_1+x_2)}_{\neq 0}=0$

$\Rightarrow \boxed{x_1=x_2}$ $\Rightarrow x_1=x_2$ on $[0,\infty)$

$y=x^2$ is 1-1 on $[0,\infty)$

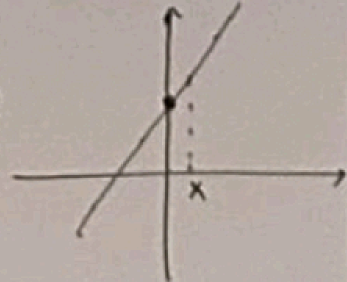
but not 1-1 on $\mathbb{R} \Rightarrow$

If f is 1-1 $\Rightarrow$ every horizontal line intersects the graph at a single pt.

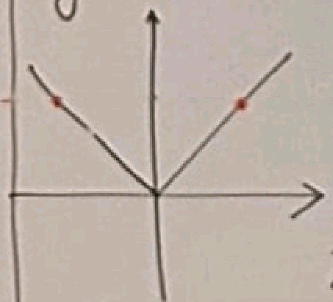
Onto function:

Let $f: A\to B$. If $\exists\, x\in A\ \forall y\in B$ s.t. $f(x)=y \Rightarrow$ f is said to be onto

e.g. $f(x)=2x+1$ onto



$g(x)=|x|$



$g: \mathbb{R} \to \mathbb{R}$, $-2 \in \mathbb{R}$ — g is not onto

$\nexists\, x\in\mathbb{R}$ s.t. $g(x)=|x|=-2$

Thm.: A function has an inverse if and only if it is one-to-one & onto.

Ex. 3-3: Find the inverse of the function

$$f(x)=\frac{x-2}{x+1}$$ on the domain $\mathbb{R}\setminus\{-1\}$

and range $\mathbb{R}\setminus\{1\}$.