$\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, \frac{1}{\infty-\infty}$

indeterminate forms

$\infty \cdot 0$

$\sqrt{x^2}=|x|=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$

$\lim_{\substack{x\to\infty \\ (x>0)}} \sqrt{x^2} = \lim_{x\to\infty} (x) = \infty$

$\lim_{\substack{x\to\infty \\ (x<0)}} \sqrt{x^2} = -\infty$

$\boxed{\lim_{x\to\infty} \left(\frac{c}{x^p}\right) = 0}$ A, $c \to$ constant, $p>0$

$p(x), q(x)$: polynomials

$\lim_{x\to\infty} \left(\frac{p(x)}{q(x)}\right) = \begin{cases} \infty & \text{if } \deg p(x) > \deg q(x) \\ * & \text{if } \deg p(x) = \deg q(x) \\ 0 & \text{if } \deg p(x) < \deg q(x) \end{cases}$

*: ratio of leading term coeffs. of the numerator & denominator.

$\ldots) = \left(\frac{1}{\sqrt{2}+1}\right)$

$\ldots + \frac{1}{x^2}) = \boxed{\infty}$

4.18) $\lim_{x\to\infty} \left(\frac{1}{2x-\sqrt{4x^2-5x+6}}\right) = \frac{1}{\infty-\infty}$?

$$= \lim_{x\to\infty} \left[\frac{(2x+\sqrt{4x^2-5x+6})}{(2x+\sqrt{4x^2-5x+6})} \cdot \frac{1}{(2x-\sqrt{4x^2-5x+6})}\right]$$

$x\sqrt{4-\frac{5}{x}+\frac{6}{x^2}}$

$$= \lim_{x\to\infty} \frac{2x+\sqrt{x^2\left(4-\frac{5}{x}+\frac{6}{x^2}\right)}}{\underbrace{\cancel{4x^2}-(\cancel{4x^2}-5x+6)}_{5x-6}}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\left[2+\sqrt{4-\frac{5}{x}+\frac{6}{x^2}}\right]}{\cancel{x}\left[5-\frac{6}{x}\right]} \quad \frac{5}{x}\to 0,\ \frac{6}{x^2}\to 0,\ \frac{6}{x}\to 0 \qquad (A)$$

$$= \frac{2+\sqrt{4-0+0}}{5-0} = \boxed{\frac{4}{5}}$$

4.16) $\lim_{x\to\infty}\left[\frac{x^4-16}{\underbrace{(2x-1)(2x+1)}(x^2+1)}\right] = \frac{\infty}{\infty} = ?$

$(4x^2-1)(x^2+1)$

$4x^4+3x^2-1$

$$= \lim_{x\to\infty} \left(\frac{x^4-16}{4x^4+3x^2-1}\right)$$

$$= \lim_{x\to\infty} \frac{\cancel{x^4}\left(1-\frac{16}{x^4}\right)}{\cancel{x^4}\left(4+\frac{3}{x^2}-\frac{1}{x^4}\right)}$$

$$= \frac{1-0}{4+0-0} = \boxed{\frac{1}{4}}$$

4.13) $\lim_{x\to\infty}\left[\frac{2+3x-4x^4}{\sqrt{x}\,(1-17x+8x^3)}\right] = \frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{(2+3x-4x^4)}{(x^{1/2}-17x^{3/2}+8x^{7/2})} = \lim_{x\to\infty} \frac{\overset{\sqrt{x}}{\cancel{x^4}}\left(\frac{2}{x^4}+\frac{3}{x^3}-4\right)}{\cancel{x^{7/2}}\left(\frac{1}{x^3}-17\frac{1}{x^2}+8\right)}$$

$$= \underbrace{\left(\lim_{x\to\infty}\sqrt{x}\right)}_{\infty}\left(\lim_{x\to\infty}\frac{\left(\frac{2}{x^4}+\frac{3}{x^3}-4\right)}{\left(\frac{1}{x^3}-\frac{17}{x^2}+8\right)}\right) = (\infty)\left(-\frac{4}{8}\right) = \boxed{-\infty}$$

$\frac{2}{x^4}\to 0,\ \frac{3}{x^3}\to 0,\ \frac{1}{x^3}\to 0,\ \frac{17}{x^2}\to 0$

4.12) $\lim_{x\to\infty}\left(\frac{3x^2+12x+9}{(x^2-1)(x^2+1)}\right) = \lim_{x\to\infty}\left(\frac{3x^2+12x+9}{x^4-1}\right)$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}\left(3+\frac{12}{x}+\frac{9}{x^2}\right)}{\underset{x^2}{\cancel{x^4}}\left(1-\frac{1}{x^4}\right)} = \underbrace{\left(\lim_{x\to\infty}\frac{1}{x^2}\right)}_{=0}\left(\lim_{x\to\infty}\frac{3+\frac{12}{x}+\frac{9}{x^2}}{1-\frac{1}{x^4}}\right) = 0\cdot\frac{3+0+0}{1-0}$$

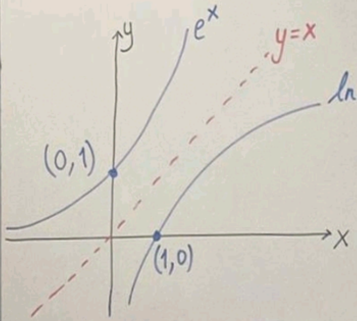
$= 0.3 = \boxed{0}$

$\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty,$

indeterminate

$\infty \cdot 0$

4-38) $\lim_{x\to\infty}\left(\frac{1}{\ln(x^2)}\right) = \frac{1}{\ln(\infty)} = \frac{1}{\infty} = \boxed{0}$



4-39) $\lim_{x\to\infty}\left(\frac{8e^x}{4+5e^x}\right) = \frac{\infty}{\infty}$

$$= \lim_{x\to\infty}\left[\frac{\cancel{e^x}(8)}{\cancel{e^x}\left(\underbrace{\frac{4}{e^x}}_{\frac{4}{\infty}=0}+5\right)}\right]$$

$$= \frac{8}{0+5} = \boxed{\frac{8}{5}}$$

4-40) $\lim_{x\to\infty}\left(\sqrt{2x^2-1}-\sqrt{x^2+1}\right) = \infty-\infty$

$$= \lim_{x\to\infty}\frac{\left(\sqrt{2x^2-1}-\sqrt{x^2+1}\right)\left(\sqrt{2x^2-1}+\sqrt{x^2+1}\right)}{\sqrt{2x^2-1}+\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty}\frac{\overbrace{(2x^2-1)-(x^2+1)}^{x^2-2}}{\sqrt{2x^2-1}+\sqrt{x^2+1}}$$

$$x \quad \underbrace{\sqrt{x^2\left(2-\frac{1}{x^2}\right)}+\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}$$

$$x\left[\sqrt{2-\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}\right]$$

$$\left(\frac{1-0}{\sqrt{2-0}+\sqrt{1+0}}\right) = \left(\frac{1}{\sqrt{2}+1}\right)$$

$$= \lim_{x\to\infty}\frac{\cancel{x^2}^{\,x}\left(1-\frac{2}{x^2}\right)}{\cancel{x}\left(\sqrt{2-\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}\right)} = \underbrace{\left(\lim_{x\to\infty} x\right)}_{\infty}\cdot\underbrace{\left(\lim_{x\to\infty}\frac{\left(1-\frac{2}{x^2}\right)}{\left(\sqrt{2-\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}\right)}\right)}_{\frac{1}{\sqrt{2}+1}} = \boxed{\infty}$$

4.31) $\lim\limits_{x\to 0}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}\right)=\dfrac{0}{0}\ ?$

$$=\lim_{x\to 0}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\right)\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$=\lim_{x\to 0}\frac{(\cancel{1}+x)-(\cancel{1}-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} \quad = 2x$$

$$=\lim_{x\to 0}\frac{2\cancel{x}}{\cancel{x}\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\frac{2}{\sqrt{1}+\sqrt{1}}=\boxed{1}$$

4.34) $\lim\limits_{x\to 1}\left(\dfrac{1-\sqrt{x}}{1-\sqrt[3]{x}}\right)=\dfrac{0}{0}$

substitution

$x=u^6 \Longrightarrow x\to 1 \Rightarrow u\to 1$

$\left.\begin{aligned}\sqrt{x}&=\sqrt{u^6}=u^3\\ \sqrt[3]{x}&=\sqrt[3]{u^6}=u^2\end{aligned}\right\}$

$$=\lim_{u\to 1}\left(\frac{1-u^3}{1-u^2}\right)=\lim_{\substack{u\to 1\\ u\neq 1}}\frac{\cancel{(1-u)}(1+u+u^2)}{\cancel{(1-u)}(1+u)}=\boxed{\frac{3}{2}}$$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$x^2-1=(x-1)(x+1)$

$x^3-1=(x-1)(x^2+x+1)$

$x^4-1=(x-1)(x^3+x^2+x+1)$

4.37) $\lim\limits_{x\to 1}\dfrac{x^n-1}{x-1}=\dfrac{0}{0}$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}\overbrace{(x^{n-1}+x^{n-2}+\cdots+x^2+x+1)}^{n\text{-terms}}}{\cancel{(x-1)}}$$

$$=\boxed{n}$$

$x^5-1=(x-1)(x^4+x^3+x^2+x+1)$

$x^n-1=(x-1)\underbrace{(x^{n-1}+x^{n-2}+\cdots+x+1)}_{n\text{-terms}}$

4.35) $\lim\limits_{x\to c}\left(\dfrac{x^4-c^4}{x^3-c^3}\right)=\dfrac{0}{0}$

$$=\lim_{\substack{x\to c\\ (x\neq c)}}\frac{\overbrace{\cancel{(x-c)}(x+c)}^{x^2-c^2}(x^2+c^2)}{\cancel{(x-c)}(x^2+xc+c^2)}=\lim_{x\to c}\left[\frac{(x+c)(x^2+c^2)}{x^2+xc+c^2}\right]$$

$$=\frac{(c+c)(c^2+c^2)}{c^2+c\,c+c^2}=\frac{(2c)(2\cancel{c^2})}{3\cancel{c^2}}=\boxed{\frac{4}{3}c}$$

4.36) $\lim\limits_{x\to 0}\left(\dfrac{x}{\sqrt{a+bx}-\sqrt{a-cx}}\right)=\dfrac{0}{0}$

$$=\lim_{x\to 0}\left(\frac{x}{\sqrt{a+bx}-\sqrt{a-cx}}\right)\left(\frac{\sqrt{a+bx}+\sqrt{a-cx}}{\sqrt{a+bx}+\sqrt{a-cx}}\right)$$

$$=\lim_{x\to 0}\frac{x\left(\sqrt{a+bx}+\sqrt{a-cx}\right)}{(\cancel{a}+bx)-(\cancel{a}-cx)}\quad (b+c)x$$

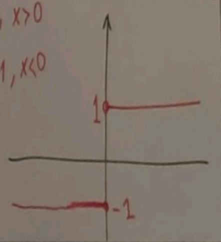
$$= \lim_{\substack{x\to0 \\ x\neq0}} \frac{\cancel{x}\left(\sqrt{a+bx}+\sqrt{a-cx}\right)}{\cancel{x}(b+c)} = \frac{\sqrt{a}+\sqrt{a}}{b+c} = \boxed{\frac{2\sqrt{a}}{b+c}}$$

4.32) $\lim_{x\to-2} \frac{(x+2)^2}{\underbrace{x^4-16}_{(x^2-4)(x^2+4)}} = \frac{0}{0}$

$$= \lim_{\substack{x\to-2 \\ x\neq-2}} \left[\frac{(x+2)\cancel{(x+2)}}{(x-2)\cancel{(x+2)}(x^2+4)}\right]$$

$$= \frac{0}{(-4)(8)} = \frac{0}{-32} = \boxed{0}$$

$$y = \frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1, & x>0 \\ \frac{-x}{x} = -1, & x<0 \end{cases}$$



4.23) $\lim_{x\to0} \frac{|x|}{x} = ?$

$x>0 \Rightarrow \lim_{x\to0} \frac{|x|}{x} = \lim_{x\to0}\left(\frac{\cancel{x}}{\cancel{x}}\right) = \lim_{x\to0}(1) = \boxed{1}$

$x<0 \Rightarrow \lim_{x\to0} \frac{|x|}{x} = \lim_{x\to0} \frac{(-x)}{\cancel{x}} = \lim_{x\to0}(-1) = \boxed{-1}$

$\Rightarrow$ $\boxed{\text{limit d.n.e.}}$ (since limits for $x>0$ and $x<0$ are not equal).

4.28) $\lim_{x\to2} \frac{x^2-7x+10}{x^2-5x+6}$ $\left(\neq \frac{1}{1} = 1\right)$ coeffs of x^2

$$= \frac{0}{0} \Rightarrow \lim_{x\to2} \frac{\cancel{(x-2)}(x-5)}{\cancel{(x-2)}(x-3)} = \frac{2-5}{2-3} = \frac{-3}{-1} = 3$$

4.27) $\lim_{x\to0} \frac{x^4-5x^2+12x+7}{5x^2+6} = \frac{0-0+0+7}{0+6} = \boxed{\frac{7}{6}}$

~~∞ (num deg > denom deg; but $x \not\to \infty$)~~