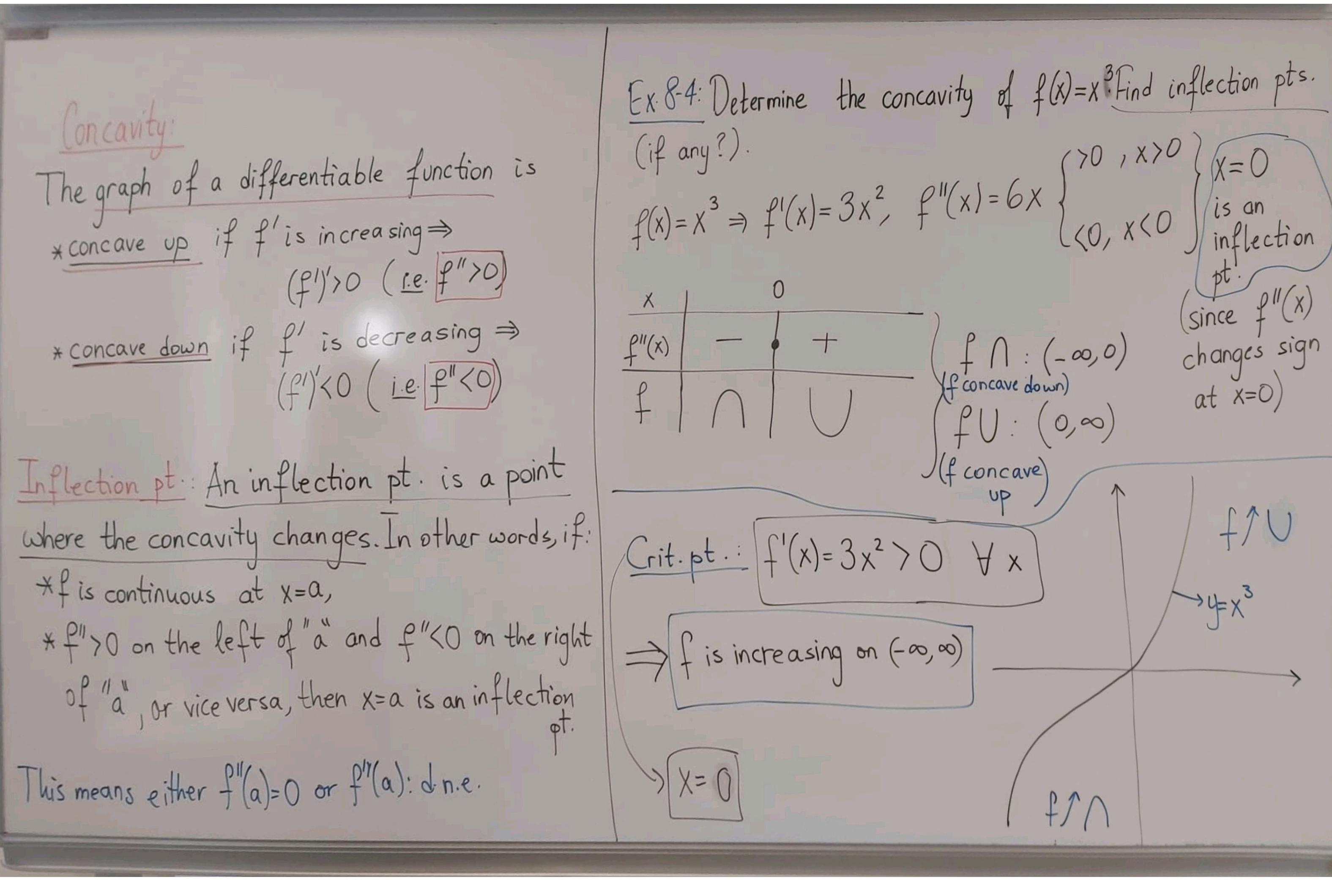


Concavity:

The graph of a differentiable function is

- * concave up if f' is increasing $\Rightarrow$ $(f')' > 0$ (i.e. $\boxed{f'' > 0}$)
- * concave down if f' is decreasing $\Rightarrow$ $(f')' < 0$ (i.e. $\boxed{f'' < 0}$)

Inflection pt.: An inflection pt. is a point where the concavity changes. In other words, if:

- * f is continuous at $x = a$,
- * $f'' > 0$ on the left of "a" and $f'' < 0$ on the right of "a", or vice versa, then $x = a$ is an inflection pt.

This means either $f''(a) = 0$ or $f''(a)$: d.n.e.

Ex. 8-4: Determine the concavity of $f(x) = x^3$. Find inflection pts. (if any?).

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2, \quad f''(x) = 6x \begin{cases} > 0, & x > 0 \\ < 0, & x < 0 \end{cases}$$

$x = 0$ is an inflection pt. (since $f''(x)$ changes sign at $x = 0$)

x		0	
$f''(x)$	$-$	●	$+$
f	$\cap$		$\cup$

$f \cap : (-\infty, 0)$ (f concave down)

$f \cup : (0, \infty)$ (f concave up)

Crit. pt.: $\boxed{f'(x) = 3x^2 > 0 \ \forall x}$

$\Rightarrow$ $\boxed{f \text{ is increasing on } (-\infty, \infty)}$

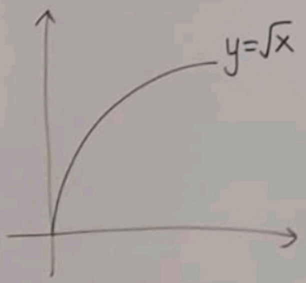
$\boxed{x = 0}$

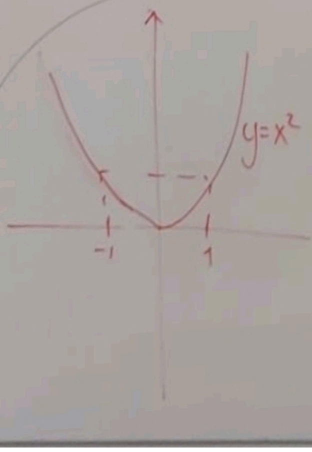
ex: $y=\sqrt{x}=(x)^{1/2}$ defined on $[0,\infty)$

$$y'=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}\begin{cases}>0 \text{ on } (0,\infty)\\ \text{d.n.e.} \\ \text{(undefined at } x=0)\end{cases}\quad \text{Crit. pt.: } \boxed{x=0}$$

$$\boxed{\Rightarrow y=\sqrt{x} \text{ is increasing on } (0,\infty)}$$

$$y''(x)=-\frac{1}{4}x^{-3/2}=-\frac{1}{4\sqrt{x^3}}\begin{cases}<0 \text{ on } (0,\infty)\Rightarrow f \cap \text{ on } (0,\infty)\\ \text{d.n.e. at } x=0\end{cases}$$





Curve sketching:

- ⊛ Identify domain of f, symmetries (odd or even function), x & y intercepts (if any)
  - odd: $f(-x)=-f(x)$
  - even: $f(-x)=f(x)$
- ⊛ Find first & second derivatives of f
- ⊛ Find $\underbrace{\text{critical pts}}_{\text{pts. where } f' \text{ changes sign}}$, $\underbrace{\text{inflection pts.}}_{\text{pts. where } f'' \text{ changes sign}}$ (if any)

  (make separate tables for f' & f'')
- ⊛ Make a table & include all the above information
- ⊛ ~~Sketch the~~ graph of the curve using the table.

$\underline{f'(x):}$

$$f'(x)=4x^3\cdot e^{-x}+x^4\cdot e^{-x}(-1)=x^3\cdot e^{-x}\left[4-x\right]$$

$$f'(x)=\frac{x^3\cdot(4-x)}{e^x}=\begin{cases}0 \Rightarrow x=0 \ \& \ x=4\\ \text{d.n.e.} \ \times \ (e^x\neq 0)\end{cases}$$

$$\left.\begin{array}{l} x=0 \Rightarrow f(0)=0 \\ x=4 \Rightarrow f(4)=(4)^4\cdot e^{-4}=\left(\frac{4}{e}\right)^4 \end{array}\right\} \begin{array}{ll}\text{crit.} & (0,0)\\ \text{pts.:} & \left(4,\left(\frac{4}{e}\right)^4\right)\end{array}$$

x		⓪		④	
f'	−	•	+	•	−
f	↘	local min	↗	local max	↘

f incr. on: $(0,4)$

f decr. on $(-\infty,0)\cup(4,\infty)$

$$f''(x)=3x^2\cdot e^{-x}(4-x)-x^3\cdot e^{-x}(4-x)-x^3\cdot e^{-x}$$

$$=x^2\cdot e^{-x}\left(12-3x-x(4-x)-x\right)=\boxed{x^2\cdot e^{-x}(x-6)(x-2)}$$

$$12-3x-4x+x^2-x$$

$$x^2-8x+12$$

$$(x-6)(x-2)$$

$\underline{\text{inf. pts:}}$ $x=0$, $x=2$, $x=6$

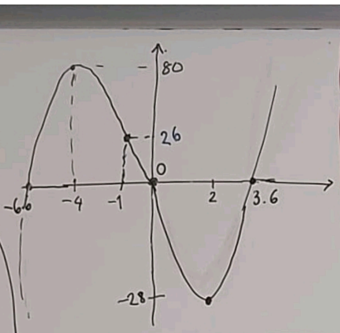
x		0		2		6	
f''	+	•	+	•	−	•	+
f	∪		∪		∩		∪

$$f(2)=2^4\cdot e^{-2}$$

$$f(6)=6^4\cdot e^{-6}$$

Ex. 8-5: $f(x)=x^3+3x^2-24x$

$\lim_{x\to\infty} f(x)=\infty$ | $\lim_{x\to-\infty} f(x)=-\infty$



Domain: $(-\infty,\infty)$

intercepts: $x^3+3x^2-24x=0$

$x(\underbrace{x^2+3x-24}_{=0})=0$

$x_1=0 \Rightarrow f(0)=0 \Rightarrow (0,0)$

$\Rightarrow x_{2,3}=\frac{-3\mp\sqrt{9-4(1)(-24)}}{2}$

$=-\frac{3}{2}\mp\frac{\sqrt{105}}{2}$ → -6.6, → 3.6

$x_2=-6.6 \Rightarrow f(-6.6)=0$

$x_3=3.6 \Rightarrow f(3.6)=0$

$f'(x)=\boxed{3x^2+6x-24}=\boxed{3(x^2+2x-8)}=0$

$3\boxed{(x+4)(x-2)}=0 \Rightarrow \boxed{x=-4,\ x=2}$

Crit. pts.:

$(2,f(2))=(2,-28)$, $(-4,f(-4))=(-4,80)$

x	-5	-4	0	2	3
f'	+	●	−	●	+
f	↗		↘		↗

↑ local max. pt. (at −4)   ↑ local min pt. (at 2)

f is incr. on: $(-\infty,-4)\cup(2,\infty)$

f is decr. on: $(-4,2)$

local max. pt.: $(-4, 80)$

local min. pt.: $(2,-28)$

$f''(x)=6x+6=0 \Rightarrow 6(x+1)=0 \Rightarrow \boxed{x=-1}$ → inf. pt.

$\Rightarrow (-1,f(-1))=\boxed{(-1,26)}$ inf. pt.

x		-1	
f''	−		+
f	∩		∪

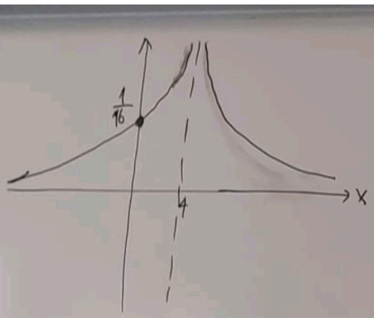
↳ inf. pt.

f is concave down on: $(-\infty,-1)$

" " " up on $(-1,\infty)$

x		-4		-1		2	
f'	+	●	−		−	●	+
f''	−		−	●	+		+
f	◜		◝		◟		◞

(8.3) $f(x)=\frac{1}{(x-4)^2}$



Domain: $\mathbb{R}\setminus\{4\}$

$(-\infty,4)\cup(4,\infty)$

$\lim_{x\to\mp\infty} f(x)=0$

$\lim_{x\to4^+} f(x)=\frac{1}{(0^+)^2}=+\infty$ | $\lim_{x\to4^-} f(x)=\frac{1}{(0^-)^2}=+\infty$

$f(0)=\frac{1}{(-4)^2}=\frac{1}{16} \Rightarrow$ y-int.: $\left(0,\frac{1}{16}\right)$

$y\neq0 \Rightarrow$ no x-int.

$f(x)=(x-4)^{-2} \Rightarrow \boxed{f'(x)=-2(x-4)^{-3}\cdot(1)}=\frac{-2}{(x-4)^3}\begin{cases}=0 \;\times\\ \text{d.n.e.}\Rightarrow x=4\end{cases}$

x	3	4	5
f'	+	‖	−
f	↗		↘

Crit. pt.: at x=4
where f is undefined

f ↗ on: $(-\infty,4)$ f ↘ on: $(4,\infty)$

$f''(x)=6(x-4)^{-4}=\frac{6}{(x-4)^4}>0 \Rightarrow$ f is $\underbrace{\text{always}}$ concave up ⟹

on: $(-\infty,4)\cup(4,\infty)$

x		4	
f'	+	‖	−
f''	+		+
f	↗		↘

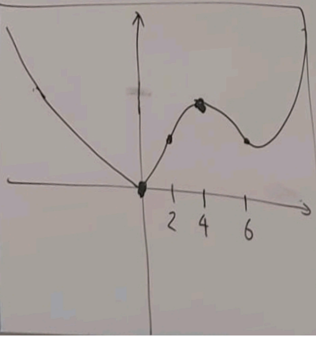
No inflection pt.

(8.6) $\boxed{f(x)=x^4\cdot e^{-x}}$

Domain: $(-\infty,\infty)$

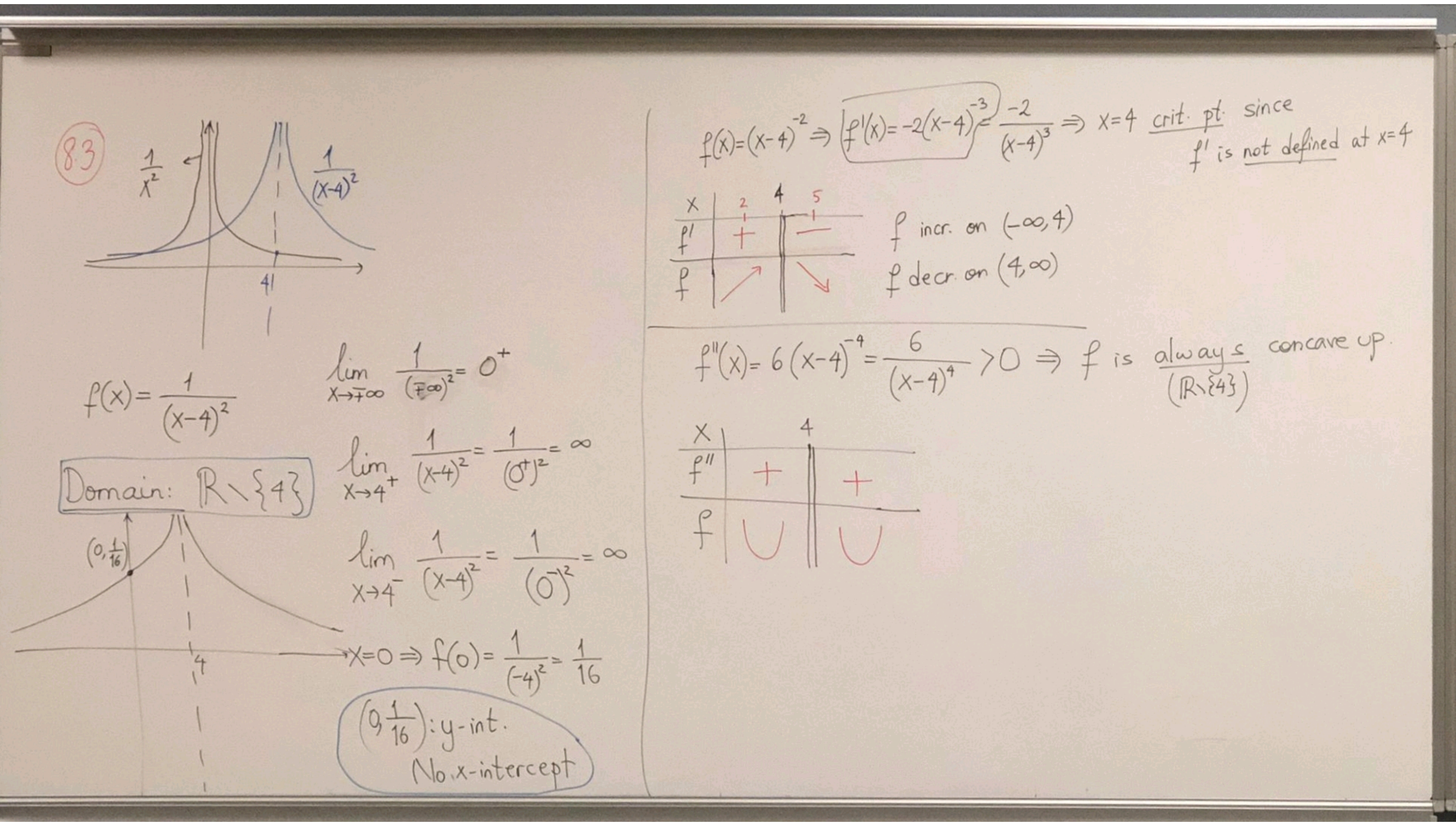
intercepts: $f(x)=\frac{x^4}{e^x} \Rightarrow x=0 \Rightarrow f(0)=0 \Rightarrow (0,0)\to$ y-intercept, x-intercept

$y=0 \Rightarrow \frac{x^4}{e^x}=0 \Rightarrow x=0$



$\lim_{x\to\infty} f(x)=0$

$\lim_{x\to-\infty} f(x)=\infty$



8.3

$$f(x) = \frac{1}{(x-4)^2}$$

$\boxed{\text{Domain: } \mathbb{R} \setminus \{4\}}$

$$\lim_{x \to \mp\infty} \frac{1}{(\mp\infty)^2} = 0^+$$

$$\lim_{x \to 4^+} \frac{1}{(x-4)^2} = \frac{1}{(0^+)^2} = \infty$$

$$\lim_{x \to 4^-} \frac{1}{(x-4)^2} = \frac{1}{(0^-)^2} = \infty$$

$$x = 0 \Rightarrow f(0) = \frac{1}{(-4)^2} = \frac{1}{16}$$

$(0, \frac{1}{16})$: y-int.

No x-intercept

$$f(x) = (x-4)^{-2} \Rightarrow \boxed{f'(x) = -2(x-4)^{-3}} = \frac{-2}{(x-4)^3} \Rightarrow x = 4 \text{ crit. pt. since}$$

f' is not defined at $x=4$

x	2	4	5
f'	+	‖	−
f	↗	‖	↘

f incr. on $(-\infty, 4)$

f decr. on $(4, \infty)$

$$f''(x) = 6(x-4)^{-4} = \frac{6}{(x-4)^4} > 0 \Rightarrow f \text{ is } \underset{(\mathbb{R} \setminus \{4\})}{\underline{\text{always}}} \text{ concave up.}$$

x		4	
f''	+	‖	+
f	∪	‖	∪

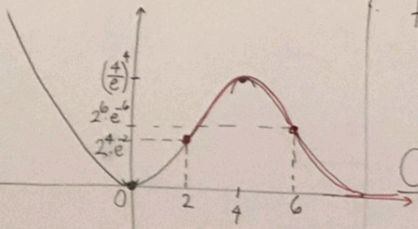
⑧.⑥ $\boxed{f(x)=x^4\cdot e^{-x}}$

Domain $f(x)$: $\mathbb{R}$

$$\left.\begin{array}{l} x=0 \Rightarrow f(0)=0 \\ f(x)=\dfrac{x^4}{e^x}=0 \Rightarrow x=0 \end{array}\right\} \begin{array}{l}(0,0)\\ x \,\&\, y\text{-int}\end{array}$$

$$\lim_{x\to\infty}\left(\frac{x^4}{e^x}\right)\to 0$$

$$\lim_{x\to-\infty}\left(x^4\cdot e^{-x}\right)=\infty$$



$$f'(x)=(4x^3)(e^{-x})+x^4(e^{-x}(-1))$$

$$=\boxed{x^3e^{-x}(4-x)}=\begin{cases}0, & x=0,\ x=4\\ \text{d.n.e.} & \times\end{cases}$$

Crit. pts: $(0,f(0)),(4,f(4))$

local min pt. $\leftarrow(0,0),\ (4,4^4\cdot e^{-4})=\left(4,\left(\frac{4}{e}\right)^4\right)$: local max pt.

x	-1	0	1	4	5
f'	−	•	+	•	−
f	↘	local min pt	↗	local max pt	↘

f incr. on: $(0,4)$

f decr. on: $(-\infty,0)\cup(4,\infty)$

$$\boxed{f''(x)}=(3x^2)(e^{-x})(4-x)+(x^3)(-e^{-x})(4-x)+x^3\cdot e^{-x}(-1)$$

$$=x^2\cdot e^{-x}[3(4-x)-x(4-x)-x]=x^2\cdot e^{-x}[12-3x-4x+x^2-x]$$

$$=\boxed{x^2}e^{-x}\underbrace{[x^2-8x+12]}_{\boxed{(x-6)(x-2)}}=0$$

$e^{-x}\neq 0$

inf pts at:
$x=0$ (double root)
$x=2$, $x=6$

$f(2)=2^4\cdot e^{-2}$, $f(6)=2^6\cdot e^{-6}$

x	-1	0	1	2	3	6	7
f''	+	•	+	•	−	•	+
f	∪		∪		∩		∪