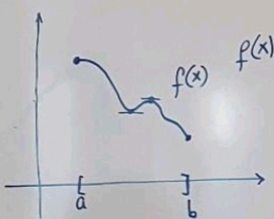
(local ↔ relative
absolute ↔ global)



$f(x)$: cont. on $[a,b]$

How to find absolute extrema:

* Find the pts. where $f' = 0$
* Find the pts. where f': d.n.e.
* Consider such pts. only if they are inside the given interval.
* Consider the value of $f(x)$ at end pts. ($x=a$ & $x=b$)
* Check all candidates ($f \to$ values). Both abs. max. & abs. min. are among them.

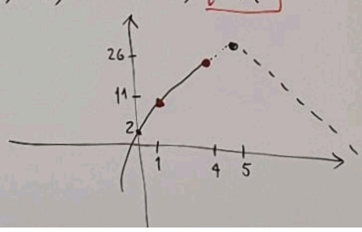
Ex. 9-1: Find the max. and min. values (i.e. abs. extrema) of

$f(x) = -x^2 + 10x + 2$ on the interval $\boxed{[1,4]}$.

* $f'(x) = -2x + 10 = 0 \Rightarrow x = \frac{10}{2} = 5 \notin [1,4] \Rightarrow$ we will not consider $x=5$ as a candidate for local extr.
* No x-value makes f': d.n.e.
* End pts: $f(1) = -1^2 + 10(1) + 2 = 11$, $f(4) = -4^2 + 10(4) + 2 = 26$

Abs. max. of f on $[1,4]$ is $\boxed{26}$ obtained at end. pt. $x=4$

Abs. min. " " " " " $\boxed{11}$ " " " " $x=1$



Ex. 9-4: $f(x)=|x-8|$ on $[6,12]$

$$\underbrace{|x-8|}_{f(x)}=\begin{cases} x-8, & x>8 \\ -(x-8), & x<8 \end{cases}$$

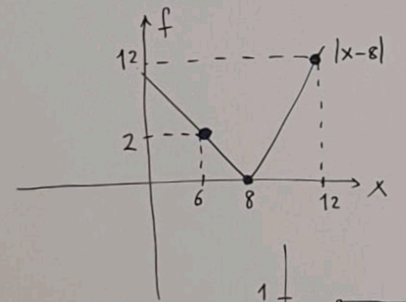
$$f'(x)=\begin{cases} 1, & x>8 \\ -1, & x<8 \end{cases} \Rightarrow f \text{ is \underline{not diff}. at } x=8$$

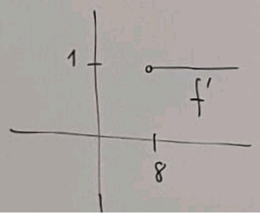
$\rightarrow$ $x=8$ is an <u>interior</u> <u>crit</u>. pt. of f on $[6,12]$

$\Rightarrow f(8)=0$

* <u>end pts:</u> $f(6)=|6-8|=2$

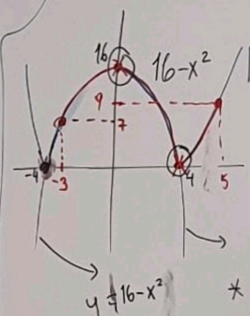
$f(12)=|12-8|=4$





x	$f(x)$
6	2
8	0 ← abs. min.
12	4 ← abs. max.

Ex. 9-5: $f(x)=|16-x^2|$ on $[-3,5]$



$$f(x)=|16-x^2|=\begin{cases} x^2-16, & x<-4 \\ 16-x^2, & -4\le x\le 4 \\ x^2-16, & x>4 \end{cases}$$

$$f'=\begin{cases} 2x, & x<-4 \\ -2x, & -4\le x\le 4 \\ 2x, & x>4 \end{cases}$$

* $f'=0 \Rightarrow x=0 \in [-3,5]$ ✓

* $f'=$ d.n.e. $\Rightarrow$ at $x=-4$ & $x=4$

	x	$f(x)$	
$f'=0$ ←	0	$\lvert 16\rvert=16$	→ abs. max. value at interior crit. pt. $f'=0$
not in $[-3,5]$; $f'=$ d.n.e.	$\cancel{-4}$	0	
$f'=$ d.n.e.	4	0	→ abs. min. value at 〃 〃 $f'=$ d.n.e.
end pts.	-3	7	
end pts.	5	9	

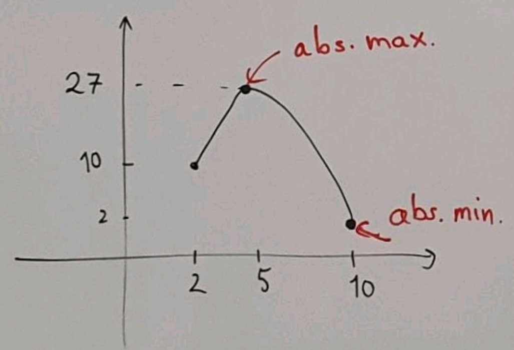
Ex. 9.2: $f(x) = -x^2 + 10x + 2$ on $[2, 10]$

$\boxed{f'(x) = 0} \Rightarrow x = 5 \in [2,10] \Rightarrow \boxed{f(5) = 27}$

	x	$f(x)$	
crit. pt. ←	5	27	→ abs. max. value
end pts.	2	18	
	10	2	→ abs. min. value

Abs. max. pt.: $(5, 27)$ (at interior crit. pt.)

Abs. min. pt.: $(2, 10)$ (at end pt.)



Ex. 9.3: $f(x) = xe^{-x}$ on $[0, 2]$

$$f'(x) = 1 \cdot e^{-x} - x \cdot e^{-x} = e^{-x}(1-x) = \frac{1-x}{\underset{\neq 0}{e^x}} = \begin{cases} 0 \Rightarrow x = 1 \in [0,2] \\ \text{d.n.e.} \Rightarrow \text{no such x-values} \end{cases}$$

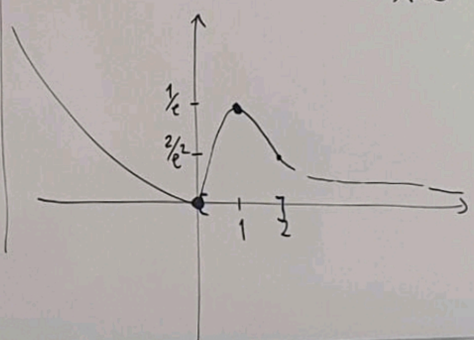
	x	$f(x)$	
crit. pt.	1	$\frac{1}{e}$	← abs. max
end pts.	0	0	← abs. min.
	2	$\frac{2}{e^2}$	

$\boxed{\frac{1}{e}} = \frac{e}{e^2} > \boxed{\frac{2}{e^2}}$

$\Rightarrow \frac{1}{e} > \frac{2}{e^2}$

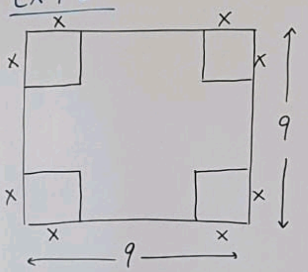
* Abs. max. of $f(x) = xe^{-x}$ is $\frac{1}{e}$ obtained at the interior crit. pt. $x = 1$

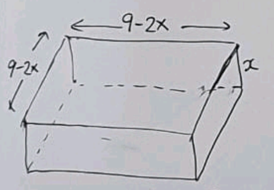
* Abs. min. of $f(x) = xe^{-x}$ is 0 obtained at the left end. pt. $x = 0$.



Applied Optimization:

Ex. 9-6:





$$\text{Volume} = \underbrace{(9-2x)^2}_{\text{base area}} \cdot \underbrace{(x)}_{\text{height}}$$

Maximize:

$$V(x) = (9-2x)^2\, x$$

where

$$0 \leqslant x \leqslant \frac{9}{2}$$

extreme values (in both cases there will be NO Box)

$$V(x) = x(81 - 36x + 4x^2)$$

$$V(x) = 4x^3 - 36x^2 + 81x$$

on $\left[0, \frac{9}{2}\right]$

$$\Rightarrow V'(x) = 12x^2 - 72x + 81 = 0$$

$$= 3\underbrace{(4x^2 - 24x + 27)}_{0} = 0$$

$$\Rightarrow (2x - 9)(2x - 3) = 0$$

$$\Rightarrow x = \frac{9}{2} \quad \text{or} \quad x = \frac{3}{2}$$

$$\Downarrow \qquad \Downarrow$$

$$V\left(\frac{9}{2}\right) = 0 \qquad V\left(\frac{3}{2}\right) = \left(9 - 2\left(\frac{3}{2}\right)\right)^2\left(\frac{3}{2}\right) = (\overset{18}{\cancel{36}})\left(\frac{3}{\cancel{2}}\right) = \boxed{54}$$

	x	$V(x)$
crit. pts.	$\frac{3}{2}$	48
end pts.	$\frac{9}{2}$	0
	0	0

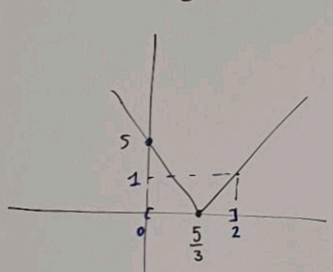
$\longrightarrow$ Max volume of the box is 54 unit^2 obtained when we cut $x = \frac{3}{2}$ units from each corner.

9-8 $f(x)=|3x-5|$ on $[0,2]$

$$=\begin{cases} 3x-5, & x>\frac{5}{3} \\ -(3x-5), & x<\frac{5}{3} \end{cases} \Rightarrow f'(x)=\begin{cases} 3, & x>\frac{5}{3} \\ -3, & x<\frac{5}{3} \end{cases}$$

$x=\frac{5}{3}$ crit. pt. where $f'(x)$: d.n.e.

& $x=\frac{5}{3}\in(0,2)$ ✓



	x	$f(x)$	
crit. pt. ←	$\frac{5}{3}$	0	⟶ abs. min. value of f is 0
end pts.	0	5	⟶ abs. max value of f is 5
	2	1	

9-10 $f(x)=x\sqrt{1-x^2}$ on $[-1,1]$.

Dom f: $[-1,1]$ (otherwise $1-x^2<0$).

$$f'(x)=(1)\sqrt{1-x^2}+(x)\cdot\frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x)$$

$$=\sqrt{1-x^2}-\frac{x^2}{\sqrt{1-x^2}}=\frac{1-x^2-x^2}{\sqrt{1-x^2}}=\frac{1-2x^2}{\sqrt{1-x^2}}\begin{cases} =0 \Rightarrow x^2=\frac{1}{2} \Rightarrow x=\mp\frac{1}{\sqrt{2}} \in(-1,1) \\ =\text{d.n.e.} \Rightarrow x=\mp 1 \text{ end. pts.} \end{cases}$$

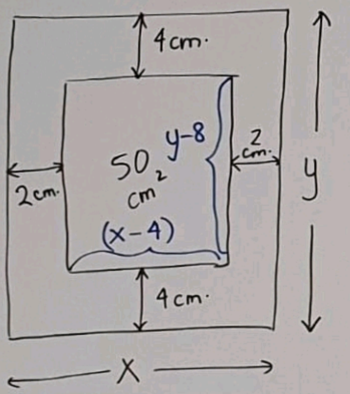
Crit. pts (x-values): $x=\mp\frac{1}{\sqrt{2}}$, $x=\mp 1$

	x	$f(x)$	
crit. pts. ($f'=0$)	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}\sqrt{\frac{1}{2}}=-\frac{1}{2}$	⟶ abs. min. value
	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\sqrt{\frac{1}{2}}=\frac{1}{2}$	→ abs. max. value
crit. pts. f': d.n.e. (also endpts)	-1	0	
	1	0	

Abs. min. value of f is $-\frac{1}{2}$ obtained at $x=-\frac{1}{\sqrt{2}}$

Abs. max. value of f is $\frac{1}{2}$ obtained at $x=\frac{1}{\sqrt{2}}$

Ex. 9.7:



Total area of the poster to be minimized! $\Rightarrow$ Minimize: $x \cdot y$

Area of the picture: $50 = (x-4)(y-8)$

$$y-8 = \frac{50}{x-4} \Rightarrow \boxed{y = \frac{50}{x-4} + 8}$$

Minimize: $f(x) = x \cdot \left(\frac{50}{x-4} + 8\right) = \frac{50x}{x-4} + 8x$,

area of the outside poster

$$f'(x) = 50\left[\frac{(1)(x-4)-(1)(x)}{(x-4)^2}\right] + 8 = \boxed{\frac{-200}{(x-4)^2} + 8 = 0}$$

$$\Rightarrow 8 = \frac{200}{(x-4)^2} \Rightarrow \cancel{8}(x-4)^2 = \cancel{200}^{25} \Rightarrow x-4 = \mp 5$$

$$x = \begin{cases} 5+4 = 9 \\ -5+4 = \cancel{-1} \end{cases}$$

$\Rightarrow$ $\boxed{x = 9 \text{ is the only crit. value}}$ $\Rightarrow$ min. value

x		0		9	
f'			$-$		$+$
f			$\searrow$		$\nearrow$
				abs. min	

$$(x-4)^2 = 25 \Rightarrow \boxed{x^2 - 8x - 9 = 0}$$

$$(x+1)(x-9) = 0$$

$$\boxed{x = 9 \Rightarrow y = 18}$$