Indefinite Integrals:

$F'(x) = f(x) \Rightarrow F$ is the antiderivative of f

(f is the derivative of F)

$\Rightarrow$ The collection of all antiderivatives of f is called the indefinite integral of f:

$$\underbrace{\int}_{\text{integration sign}} f(x)\, d\underbrace{x}_{\text{integration variable}} = F(x) + \underbrace{\overbrace{C}^{\text{arbitrary constant}}}_{\text{integration constant}}$$

$$\left.\begin{aligned}\int x^2 dx &= \frac{x^3}{3} + 5 \\ \int x^2 dx &= \frac{x^3}{3} - 10000 \\ &\vdots\end{aligned}\right\} \boxed{\int x^2 dx = \frac{x^3}{3} + C}$$

$$\int x = ?$$

$$\int x\, dx = \frac{x^2}{2} + C$$

Some basic indefinite integrals (using derivative formulae):

$$\int (1)\, dx = x + C \Rightarrow \int a\, dx = ax + c \quad (a \to \text{constant})$$

$$\int (x)\, dx = \frac{x^2}{2} + C \quad \left(\text{since; } \frac{d}{dx}\left(\frac{x^2}{2}\right) = x\right).$$

$$n \in \mathbb{R} \setminus \{-1\} \quad \boxed{\int (x^n)\, dx = \frac{x^{n+1}}{n+1} + C} \quad (n \neq -1)$$

$$\boxed{\int \left(\frac{1}{x}\right) dx = \ln|x| + C}$$

$$\int (e^x)\, dx = e^x + C$$

$$\boxed{\int (e^{ax})\, dx = \frac{e^{ax}}{a} + C}$$

Ex 10.1: Evaluate the integral $\int\left(\frac{1}{x^3} - 2x + 4\right)dx = ?$

$$\Rightarrow \int(x^{-3} - 2x + 4)\,dx = \int x^{-3}dx - 2\int x\,dx + 4\int 1\,dx$$

$$= \boxed{\frac{x^{-3+1}}{-3+1} - \cancel{2}\,\frac{x^2}{\cancel{2}} + 4x + C}$$

$$= \boxed{-\frac{1}{2x^2} - x^2 + 4x + C}$$

Ex 10.2: $\int\left(\frac{x^2-1}{x\sqrt{x}}\right)dx = ?$

$$= \int \frac{x^2}{x^{3/2}}\,dx - \int x^{-3/2}\,dx$$

$$= \int x^{1/2}\,dx - \int x^{-3/2}\,dx = \frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{-1/2}}{-\frac{1}{2}} + C$$

$$= \boxed{\frac{2}{3}\,x\sqrt{x} + 2\,\frac{1}{\sqrt{x}} + C}$$

Ex 10-3: Find a function $f(x)$ such that

$f'(x) = 5e^x$ and $\boxed{f(0) = 9}$ → initial condition.

$$\boxed{\int f'(x)\,dx = f(x) + C}$$

$$\underbrace{\int f'(x)\,dx}_{f(x)} = \int 5e^x\,dx = 5e^x + C$$

$$\Rightarrow \boxed{f(x) = 5e^x + C} \Rightarrow \underbrace{f(0)}_{\boxed{9}} = 5\underbrace{e^0}_{1} + C = \boxed{5 + C}$$

$$\Rightarrow 5 + C = 9 \Rightarrow \boxed{C = 4}$$

$$\Rightarrow \text{Ans: } \boxed{f(x) = 5e^x + 4}$$

10-4: Find a function $f(x)$ such that

$f''(x) = 4 - \frac{8}{x^2}$ and $\boxed{f(1) = -15}$, $\boxed{f'(1) = 7}$ ✓

$$\underbrace{\int f''(x)\,dx}_{f'(x)} = \int (4 - 8x^{-2})\,dx = 4x - 8\frac{x^{-1}}{-1} + C_1$$

$$\boxed{f'(x) = 4x + \frac{8}{x} + C_1}, \quad \underbrace{f'(1)}_{\boxed{7}} = 4(1) + \frac{8}{(1)} + C_1 = \boxed{12 + C_1}$$

$$\Rightarrow \boxed{C_1 = -5}$$

$$\Rightarrow \boxed{f'(x) = 4x + 8x^{-1} - 5}$$

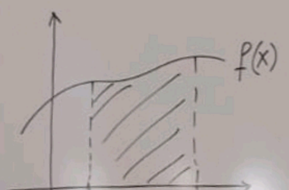
$$\underbrace{\int f'(x)\,dx}_{f(x)} = \int (4x + 8x^{-1} - 5)\,dx = \boxed{4\frac{x^2}{2} + 8\ln|x| - 5x + C_2}$$

$$\Rightarrow f(x) = 2x^2 + 8\ln|x| - 5x + C_2$$

$$\underbrace{f(1)}_{-15} = 2 + 8\underbrace{\ln|1|}_{0} - 5(1) + C_2 = -3 + C_2 \Rightarrow C_2 = -15 + 3 = \boxed{-12}$$

$$\Rightarrow \boxed{f(x) = 2x^2 + 8\ln|x| - 5x - 12}$$

Fundamental Theorem of Calculus:



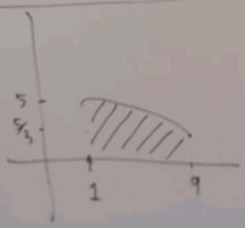
$$\boxed{\int_a^b f(t)\,dt = F(b) - F(a)}$$

$\begin{cases} f \text{ cont. on } [a,b] \\ F\text{: any antideriv. of } f \end{cases}$

$\int_a^b$: definite integral of f over $[a,b]$.

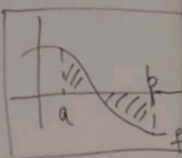
We will use the notation: $F(t)\Big|_a^b = F(b) - F(a)$

Ex. 10-5: Evaluate $\int_1^9 \frac{5}{\sqrt{x}}\,dx = ?$



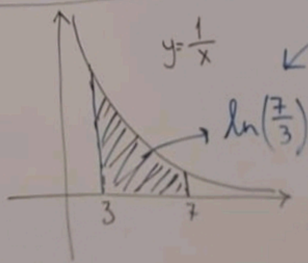
$$\int_1^9 \frac{5}{\sqrt{x}}\,dx = \left(10\sqrt{x}\right)\Big|_{x=1}^{x=9} = 10\left(\sqrt{9} - \sqrt{1}\right) = 10\,(3-1) = \boxed{20}$$

$$\int \frac{5}{\sqrt{x}}\,dx = \int 5x^{-1/2}\,dx = 5\cdot\frac{x^{1/2}}{1/2} + C = \underbrace{10\sqrt{x}}_{F(x)} + C$$



Ex. 10-6: Evaluate $\int_3^7 \frac{dx}{x} = ?$

$$\int_3^7 \frac{dx}{x} = \ln|x|\Big|_3^7 = \ln 7 - \ln 3 = \boxed{\ln\left(\frac{7}{3}\right)}$$



$\int_3^7 \frac{dx}{x}$: area under the curve $y = \frac{1}{x}$ from $x=3$ to $x=7$

Evaluate the following definite integrals:

10-16) $\int_{-1}^{2}(1+4e^{x})\,dx = (x+4e^{x})\Big|_{-1}^{2}$

$=(2+4e^{2})-((-1)+4e^{-1})$

$=3+4\left(e^{2}-\frac{1}{e}\right)$

$=\boxed{3+4\left(\frac{e^{3}-1}{e}\right)}$

10-18) $\int_{1}^{9}\left(\frac{1-\sqrt{x}}{\sqrt{x}}\right)dx = \int_{1}^{9}\left(x^{-1/2}-1\right)dx$

$=\left(\frac{x^{1/2}}{\frac{1}{2}}-x\right)\Big|_{1}^{9} = 2(3-1)-(9-1)$

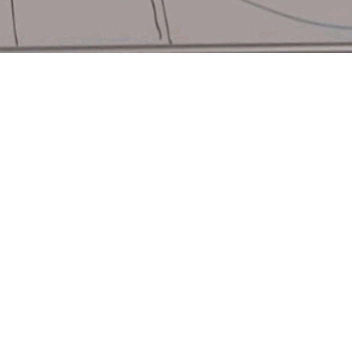
$=4-8=\boxed{-4}$

10-19) $\int x^{-2/5}\,dx \underset{\substack{\downarrow \\ \text{indefinite} \\ \text{int.}}}{=} \frac{x^{-2/5+1}}{-2/5+1}+C=\boxed{\frac{5}{3}x^{3/5}+C}$

$\int_{1}^{32} x^{-2/5}\,dx = \frac{5}{3}\big(\underbrace{x^{3/5}}_{(x^{1/5})^{3}}\big)\Big|_{1}^{32} = \frac{5}{3}\left\{\big[\underbrace{(32)^{1/5}}_{}\big]^{3} - \big[\underbrace{1^{3/5}}_{=1}\big]\right\} = \frac{5}{3}\left(2^{3}-1\right)=\boxed{\frac{35}{3}}$

$\big(\underbrace{(2^{5})^{1/5}}_{2}\big)^{3}$

10-20) $\int_{-2}^{-1}\frac{1}{x^{3}}\,dx = \int_{-2}^{-1}x^{-3}\,dx = \frac{x^{-2}}{-2}\Big|_{-2}^{-1} = -\frac{1}{2}\,\frac{1}{x^{2}}\Big|_{-2}^{-1}$



$=-\frac{1}{2}\left[\frac{1}{(-1)^{2}}-\frac{1}{(-2)^{2}}\right]=-\frac{1}{2}\left[1-\frac{1}{4}\right]$

$=-\frac{1}{2}\left(\frac{3}{4}\right)=\boxed{-\frac{3}{8}}$