Find $\boxed{y'}$ using implicit differentiation:

$y = y(x) \Rightarrow \frac{d}{dx}(y(x)) = y'(x)$

7.4) $\boxed{x = y + y^{2/3}}$

$$\frac{d}{dx}(x) = \frac{d}{dx}(y) + \frac{d}{dx}\left(y^{2/3}\right) \qquad y \downarrow x$$

chain rule

$$1 = y' + \left(\frac{2}{3} y^{\frac{2}{3}-1} \cdot y'\right)$$

$$1 = y'\left[1 + \frac{2}{3} y^{-1/3}\right] = y'\left[1 + \frac{2}{3\sqrt[3]{y}}\right]$$

$$\Rightarrow \boxed{y'} = \frac{1}{\frac{3\sqrt[3]{y}+2}{3\sqrt[3]{y}}} = \boxed{\frac{3\sqrt[3]{y}}{3\sqrt[3]{y}+2}}$$

$$\boxed{y' = \frac{3y^{1/3}}{3y^{1/3}+2}}$$

7.7 $e^{xy} = x + 2y \Rightarrow \frac{d}{dx}\left(e^{xy}\right) = \frac{d}{dx}(x) + \frac{d}{dx}(2y)$

$$e^{xy} \cdot \left[\underbrace{(1)(y) + (x)(y')}_{\frac{d}{dx}(xy)}\right] = 1 + 2y'$$

$$y'\left[x \cdot e^{xy} - 2\right] = 1 - y \cdot e^{xy} \Rightarrow \boxed{y' = y'(x) = \frac{1 - ye^{xy}}{xe^{xy} - 2}}$$

7.9 $y^2 \cdot \ln y = x^3 \cdot e^x \qquad y = y(x) \Rightarrow y' = y'(x)$

$$\left[(2y \cdot y')(\ln y) + (y^2)\left(\frac{1}{y} \cdot y'\right)\right] = (3x^2)(e^x) + (x^3)(e^x)$$

$$y'\Big|_{(1,5)} = \frac{-4(1-3(5))\sqrt{11+5^2}}{5-(12(1)-4(5))\sqrt{11+5^2}} = \frac{-4(-14)(6)}{5-(-8)(6)}$$

$$= \boxed{\frac{336}{53}} = m \text{ : slope of tg. line}$$

tg. line eqn.: $\boxed{y-5 = \left(\frac{336}{53}\right)(x-1)}$

7.18

$$\ln(xy) + xy^2 - \ln(3x) - 6y = 0 \quad \text{at } (2,3)$$

$$\frac{\overbrace{\frac{d}{dx}(xy)}^{1.y+x.y'}}{xy} + [1.y^2 + x.2y.y'] - \frac{\cancel{3}}{\cancel{3}x} - 6y' = 0$$

$$y'\left[\frac{\cancel{x}}{\cancel{x}y} + 2xy - 6\right] = \frac{1}{x} - y^2 - \frac{\cancel{y}}{x\cancel{y}}$$

$$\boxed{y' = \frac{\frac{1}{\cancel{x}} - y^2 - \frac{1}{\cancel{x}}}{\frac{1}{y} + 2xy - 6}}$$

$$y'\Big|_{(2,3)} = \frac{-3^2}{\frac{1}{3} + 2(2)(3) - 6}$$

$$y'\Big|_{(2,3)} = \frac{-9}{\frac{1+18}{3}} = \boxed{-\frac{27}{19}}$$

7.17 $xe^x - ye^y + xy = 1$ at $(1,1)$

$$(1.e^x + x.e^x) - (y'.e^y + y(e^y.y')) + (1.y + x.y') = 0$$

$$y'[-e^y - ye^y + x] = y - e^x - xe^x$$

$$\boxed{y' = \frac{y - e^x - xe^x}{x - e^y - ye^y}}$$

$$y'\Big|_{(1,1)} = \frac{1-e-e}{1-e-e} = \frac{1-2e}{1-2e} = 1$$

tg. line: $y - 1 = (1)(x-1) \rightarrow \boxed{y = x}$

**Ex. 7.6:** $\lim\limits_{x\to 1} \dfrac{x^{10}-1}{x^{7}-1} = \left(\dfrac{0}{0}\right)$

$$x^{10}-1 = (x-1)(x^9 + x^8 + x^7 + \cdots + 1)$$
$$x^{7}-1 = (x-1)(x^6 + x^5 + \cdots + 1)$$

$$\overset{L.H}{=} \lim_{x\to 1} \frac{10x^9}{7x^6} = \frac{10(1)}{7(1)} = \boxed{\frac{10}{7}}$$

$$\begin{array}{l|l} x^{10}-1 & x-1 \\ \hline -x^{10}+x^9 & x^9+x^8+x^7+\cdots+1 \\ \hline x^9-1 & \\ -x^9+x^8 & \\ \hline x^8-1 & \end{array}$$

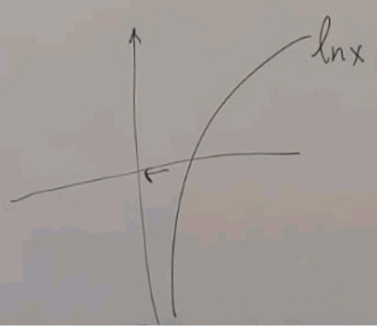
**Ex. 7.7:** Evaluate the limit (if it exists)

$$\lim_{x\to 0} \frac{e^x-1-x-\frac{x^2}{2}}{x^3+x^4} = \left(\frac{0}{0}\right)$$

$$\overset{L.H.}{=} \lim_{x\to 0} \frac{e^x-1-\frac{2x}{2}}{3x^2+4x^3} = \left(\frac{0}{0}\right)$$

$$\overset{L.H.}{=} \lim_{x\to 0} \frac{e^x-1}{6x+12x^2} = \left(\frac{0}{0}\right)$$

$$\overset{L.H.}{=} \lim_{x\to 0} \frac{e^x}{6+24x} = \frac{1}{6+0} = \frac{1}{6}$$




**Ex. 7.8:**

$$\lim_{x\to\infty} \frac{\ln x + x^2}{x e^x} = \frac{\infty}{\infty}$$

$$\overset{L.H.}{=} \lim_{x\to\infty} \frac{\frac{1}{x}+2x}{\underbrace{1\cdot e^x + x\cdot e^x}_{(1+x)e^x}} = \frac{\infty}{\infty}$$

$$\overset{L.H.}{=} \lim_{x\to\infty} \frac{-\frac{1}{x^2}+2}{\underbrace{1\cdot e^x+(1+x)e^x}_{(2+x)e^x}} = \frac{0+2}{\infty} = \frac{2}{\infty} = \boxed{0}$$

**7.36** $\lim\limits_{x\to 0^+} (x\cdot \ln x) = 0\cdot(-\infty)$

$$= \lim_{x\to 0^+} \left(\frac{\ln x}{\frac{1}{x}}\right) = \frac{-\infty}{\infty} \text{ (indeterminate form)}$$

$$\overset{L.H.}{=} \lim_{x\to 0^+} \left(\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right) = \lim_{x\to 0^+} (-x) = \boxed{0}$$

## L'Hôpital's rule:

$\frac{0}{0}, \frac{\infty}{\infty}, (\infty - \infty)$: indeterminate forms

**Ex. 7.4:** $\frac{0}{0}$ limit ?

* $\lim\limits_{x\to 0} \frac{x^5}{x^2} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 0}(x^3) = \boxed{0}$ ✓
* $\lim\limits_{x\to 0}\left(\frac{x^5}{x^7}\right) = \lim\limits_{x\to 0}\left(\frac{1}{x^2}\right) = \boxed{\infty}$ ✓
* $\lim\limits_{x\to 0}\left(\frac{3x^5}{4x^5}\right) = \boxed{\frac{3}{4}}$ ✓

## L'Hôpital's rule:

Consider the limit: $\boxed{\lim\limits_{x\to a} \frac{f(x)}{g(x)}}$

Assume that we have an indeterminate form of the type: $\boxed{\frac{0}{0} \text{ or } \frac{\infty}{\infty}}$

Suppose $g'(x) \neq 0$ on an open interval containing $x = a$

$g'(x) \neq 0$

$a$

Then: $\boxed{\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}}$ if this limit exists or is $\mp\infty$.

**Ex. 7.5:** $\lim\limits_{x\to\infty} \frac{e^x}{x^3} = \frac{\infty}{\infty}$

L'Hôpital's

$$\overset{\text{L.H.}}{=} \lim_{x\to\infty} \frac{e^x}{3x^2} \underset{\frac{\infty}{\infty}}{\overset{\text{L.H.}}{=}} \lim_{x\to\infty} \frac{e^x}{6x} \underset{\frac{\infty}{\infty}}{\overset{\text{L.H.}}{=}} \lim_{x\to\infty} \frac{e^x}{6} = \frac{\infty}{6} = \boxed{\infty}$$

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \frac{\overset{n>0}{\text{const}}}{\infty} = \boxed{0}$$

Exp. func. grows faster than any polynomial!

$$y'[\underbrace{2y\cdot \ln y + y}_{y[2\ln y+1]}] = x^2 e^x(3+x)$$

$$\Rightarrow \boxed{y' = \frac{x^2e^x(3+x)}{y(2\ln y+1)}}$$

7.12 $x^{1/3}+y^{1/5}=y$

$$\frac{1}{3}x^{-2/3} + \frac{1}{5}y^{-\frac{4}{5}}\cdot y' = y'$$

$$\frac{1}{3x^{2/3}} = y'\left[1-\frac{1}{5y^{4/5}}\right]$$

$$\frac{1}{3x^{2/3}} = y'\left[\frac{5y^{4/5}-1}{5y^{4/5}}\right]$$

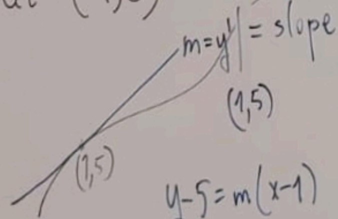
$$\boxed{y' = \frac{5y^{4/5}}{3x^{2/3}\cdot[5y^{4/5}-1]}}$$

Find $y'$ at the indicated pt. and then find the tg. line eqn. to the curve at the given pt.

7.16

$$\sqrt{11+y^2}-12xy+2y^2+4x=0$$

at $(1,5)$




$$y-5=m(x-1)$$

$$(11+y^2)^{1/2}-12xy+2y^2+4x=0$$

$$\frac{1}{\cancel{2}}(11+y^2)^{-1/2}\cdot(0+\cancel{2}y\cdot y') - 12[(1)(y)+(x)\cdot(y')]+4y\cdot y'+4=0$$

$$\frac{y\cdot y'}{\sqrt{11+y^2}} - 12[y+xy'] + 4y\cdot y'+4=0$$

$$y'\left[\frac{y}{\sqrt{11+y^2}} - 12x+4y\right] = -4+12y$$

$$y' = \frac{-4+12y}{\frac{y-(12x-4y)\sqrt{11+y^2}}{\sqrt{11+y^2}}}$$

$$y' = \frac{-4(1-3y)\sqrt{11+y^2}}{y-(12x-4y)\sqrt{11+y^2}}$$