Evaluate the following limits:

L'Hôpital's rule: L.H. $\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$

7.22 $\lim_{x\to\infty}\left[\frac{3x^2+4\ln x}{6x^2+7\ln x}\right] = \frac{\infty}{\infty}$

$$\overset{L.H}{\Longrightarrow} = \lim_{x\to\infty}\left[\frac{6x+4\cdot\frac{1}{x}}{12x+7\cdot\frac{1}{x}}\right] = \left(\frac{\infty}{\infty}\right)$$

$$\overset{L.H.}{=} \lim_{x\to\infty}\left(\frac{6-\frac{4}{x^2}}{12-\frac{7}{x^2}}\right) = \frac{6-0}{12-0} = \boxed{\frac{1}{2}}$$

or

$$= \lim_{x\to\infty}\left(\frac{6x^2+4}{12x^2+7}\right) = \left(\frac{\infty}{\infty}\right)$$

$$\overset{L.H.}{=} \lim_{x\to\infty}\left(\frac{12x}{24x}\right) = \boxed{\frac{1}{2}}$$

$$[\ln(u(x))]' = \frac{u'(x)}{u(x)}$$

7.24 $\lim_{x\to\infty}\left(\frac{\ln(x+x^4)}{x}\right) = \left(\frac{\infty}{\infty}\right)$

$$\overset{L.H.}{=} \lim_{x\to\infty}\frac{\frac{1+4x^3}{x+x^4}}{1} = \lim_{x\to\infty}\frac{\cancel{x^3}\left(\frac{1}{x^3}+4\right)}{\cancel{x^3}\left(\frac{1}{x^2}+x\right)} = \frac{0+4}{0+\infty} = \frac{4}{\infty} = \boxed{0}$$

7.27 $\lim_{x\to\frac{1}{2}}\left(\frac{\ln(2x)}{2x^2+x-1}\right) = \frac{\ln 1}{\frac{1}{2}+\frac{1}{2}-1} = \left(\frac{0}{0}\right) \Longrightarrow$ use L.H

$$\overset{L.H.}{=} \lim_{x\to\frac{1}{2}}\left(\frac{\left(\frac{\cancel{2}}{\cancel{2}x}\right)}{4x+1}\right) = \frac{1}{\left(\frac{1}{2}\right)\left(4\left(\frac{1}{2}\right)+1\right)} = \boxed{\frac{2}{3}}$$

7.28 $\lim_{x\to e}\left(\frac{\ln x-1}{x-e}\right) = \left(\frac{0}{0}\right)$ $(\ln e = 1)$

$$\overset{L.H.}{=} \lim_{x\to e}\left(\frac{\frac{1}{x}}{1}\right) = \boxed{\frac{1}{e}}$$

$$\left[(a+bx)^{1/2}\right]' = \frac{1}{2}(a+bx)^{-1/2}\cdot b$$

7.31 $\lim_{x\to\infty}\left(\frac{\ln x}{\sqrt[3]{x}}\right)=\left(\frac{\infty}{\infty}\right)$

$$\overset{L.H.}{=}\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}}=3\lim_{x\to\infty}\left(\frac{x^{2/3}}{x}\right)=3\lim_{x\to\infty}\left(\frac{1}{x^{1/3}}\right)$$

$$=3\cdot\frac{1}{\infty}=\boxed{0}$$

7.33 $\lim_{x\to 0}\left(\frac{\sqrt{a+bx}-\sqrt{a+cx}}{x}\right)=\frac{0}{0}$

$$\overset{L.H.}{=}\lim_{x\to 0}\frac{\frac{b}{2\sqrt{a+bx}}-\frac{c}{2\sqrt{a+cx}}}{1}$$

$$=\frac{b-c}{2\sqrt{a}}$$

7.35 $\lim_{x\to 2}\left[\frac{\ln\left(\frac{x}{2}\right)}{\underbrace{x(x-2)}_{x^2-2x}}\right]=\frac{0}{0}\Rightarrow$ use L.H.

$$\overset{L.H}{=}\lim_{x\to 2}\frac{\frac{\frac{1}{2}}{\left(\frac{x}{2}\right)}}{2x-2}=\lim_{x\to 2}\frac{1}{2x(x-1)}=\frac{1}{2(2)(2-1)}=\boxed{\frac{1}{4}}$$

$y=4^x\Rightarrow y'=?$ log. diff.

$$\ln y=\ln 4^x=x\cdot\ln 4\underset{\left(\frac{d}{dx}\right)}{\Rightarrow}\boxed{\frac{y'}{y}}=(\ln 4)(1)$$

$$\Rightarrow\boxed{y'=y\ln 4=4^x\ln 4}$$

$$\frac{d}{dx}(e^x)=e^x\cdot\overbrace{\ln e}^{1}=e^x$$

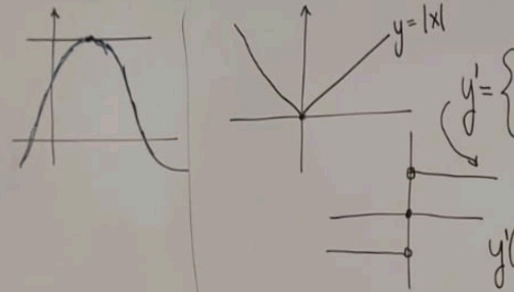
$$\frac{d}{dx}(a^x)=a^x\cdot\ln a$$

7.34 $\lim_{x\to 0}\left(\frac{4^x-1}{2^x-1}\right)=\left(\frac{0}{0}\right)\overset{L.H.}{\Rightarrow}=\lim_{x\to 0}\frac{(4^x)(\ln 4)}{(2^x)(\ln 2)}=\boxed{\frac{\ln 4}{\ln 2}}=\frac{\ln 2^2}{\ln 2}=2$

$\neq\ln(4-2)=\ln 2$

First Derivative Test:

Critical pt.: at a crit. pt., the derivative is zero or undefined.



$$y' = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$$

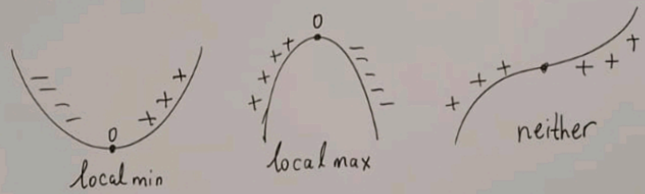
$y'(0) = \text{undef.}$

Let f be a continuous func. and let $x=c$ be a critical point of it. Suppose f' exists in some interval containing $x=c$, except possibly at $x=c$.

f has a local extremum at "c" if and only if f' changes sign at "c".

* sign change of f' $-$ to $+$ $\Rightarrow$ $f(c)$ is a local min. $((c, f(c)) \to$ local min. pt.$)$
* sign change of f' $+$ to $-$ $\Rightarrow$ $f(c)$ " " local max. $((c, f(c)) \to$ local max. pt.$)$

f':



Ex 8.1: Find the intervals where $\boxed{f(x) = 2x^3 - 9x^2 + 5}$ is increasing ($f' > 0$) and decreasing ($f' < 0$). Then find all the local extrema of this function using first der. test.

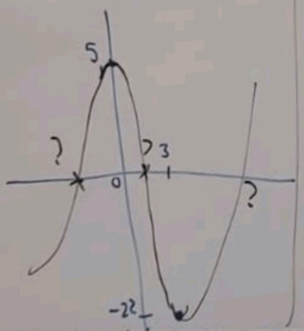
Soln.: $f(x) = 2x^3 - 9x^2 + 5 \Rightarrow f'(x) = 6x^2 - 18x$

$$\Rightarrow f'(x) = 6x(x-3) = \begin{cases} = 0 \Rightarrow \boxed{x=0,\ x=3} \\ = \text{undefined} \ \times \end{cases}$$

$\boxed{\text{Crit. pts.:}}$ $\boxed{x=0,\ x=3}$

$(0, f(0)) = \boxed{(0,5)}$

$(3, f(3)) = \boxed{(3,-22)}$



f' changes sign at $\underbrace{(0,5) \text{ and at } (3,-22)}$ $\Rightarrow$ by first der. test there are local extrema at these crit. pt.

x	-1	0	1	3	5
$f' = x(x-3)$	$+$	•	$-$	•	$+$
f	↗ incr.	local max. pt.	↘ decr.	local min. pt.	↗ incr.

f incr.: $(-\infty, 0) \cup (3, \infty)$

f decr.: $(0, 3)$

local max. pt(s): $(0,5)$

local min. pt(s): $(3,-22)$

Ex. 8-2: Find the intervals where $f(x)=x^{4/5}$ is increasing and decreasing. Then find all the local extrema of this function using the first der. test.

$$f(x)=x^{4/5} \Rightarrow f'(x)=\frac{4}{5}x^{-1/5}=\frac{4}{5\sqrt[5]{x}}$$

$$f'(x)=\frac{4}{5\sqrt[5]{x}}=\begin{cases} 0 & \times \\ \text{undefined} \Rightarrow \boxed{x=0} \end{cases}$$

$$\Rightarrow \boxed{(0,f(0))=(0,0) \to \text{crit. pt.}}$$

x	-1	0	1 ← test x-values
f'	−		+
f	↘		↗

f incr. (↗): $(0,\infty)$
f decr. (↘): $(-\infty,0)$

$(0,0)$ is a local min. pt. (by first der. test)

Ex. 8-3: $\boxed{f(x)=x^4-4x^3-2x^2+12x} \Rightarrow f'(x)=4x^3-12x^2-4x+12$
$=4[x^3-3x^2-x+\underset{\mp1,\mp3}{3}]=0$

check the roots of $\boxed{x^3-3x^2-x+3=0}$ at $\mp1, \mp3$

$x=1$ is a root $\Rightarrow$

$$\begin{array}{l|l} x^3-3x^2-x+3 & x-1 \\ \underline{-x^3 \pm x^2} & x^2-2x-3=(x-3)(x+1) \\ -2x^2-x+3 & \\ \underline{\pm 2x^2 \mp 2x} & \\ -3x+3 & \\ \underline{-3x+3} & \\ 0 \quad 0 & \end{array}$$

$f'(x)=4(x^3-3x^2-x+3)$
$=4(x-1)(x-3)(x+1)=0$

Crit. pts: $(1,f(1))=(1,7)$ local max
$(3,f(3))=(3,-9)$ local min
$(-1,f(-1))=(-1,-9)$ local min

x	-2	-1	0	1	2	3	4
f'	−		+		−		+
f	↘		↗		↘		↗
		local min		local max		local min	

Int(s) of increase: $(-1,1)\cup(3,\infty)$
" " decrease: $(-\infty,-1)\cup(1,3)$