Logarithmic Differentiation:

Remember: $\left[\frac{d}{dx}(\ln u(x)) = \frac{u'(x)}{u(x)}\right]$

$\log_a A^r = r \log_a A$

Logarithm transforms product into sums.

$(\log_a (AB) = \log_a A + \log_a B)$

This helps in finding derivatives of complicated functions.

For example; $y = \frac{(x^3+1)(x^2-1)}{x^8+6x^4+1}$, $y' = ?$

$y = y(x)$

$\Rightarrow \ln y = \ln(x^3+1) + \ln(x^2-1) - \ln(x^8+6x^4+1)$

Diff. both sides w.r.t. "x": using chain rule $\Rightarrow$

$$\frac{y'}{y} = \frac{3x^2}{x^3+1} + \frac{2x}{x^2-1} - \frac{8x^7+24x^3}{x^8+6x^4+1} \quad \text{(logarithmic diff.)}$$

$$\Rightarrow y' = y\left[\frac{3x^2}{x^3+1} + \frac{2x}{x^2-1} - \frac{8x^7+24x^3}{x^8+6x^4+1}\right]$$

$$\Rightarrow y'(x) = \left[\frac{(x^3+1)(x^2-1)}{x^8+6x^4+1}\right]\left[\frac{3x^2}{x^3+1} + \frac{2x}{x^2-1} - \frac{8x^7+24x^3}{x^8+6x^4+1}\right]$$

Ex. 6-10: Find the derivative of the function

$$y = f(x) = (x+e^x)^{\ln x} \qquad [y = (u(x))^{v(x)}]$$

$$\ln y = (\ln x)[\ln(x+e^x)]$$

Keeping in mind that $y = y(x)$ and using both logarithmic diff. and the product rule we get:

$$\frac{y'}{y} = \left(\frac{1}{x}\right)\left[\ln(x+e^x)\right] + (\ln x)\left[\frac{1+e^x}{x+e^x}\right]$$

$$\Rightarrow y' = y\left[\frac{\ln(x+e^x)}{x} + \frac{(\ln x)(1+e^x)}{(x+e^x)}\right]$$

$$\boxed{y' = (x+e^x)^{\ln x} \cdot \left[\frac{\ln(x+e^x)}{x} + \frac{(\ln x)(1+e^x)}{(x+e^x)}\right]}$$

Find f' using logarithmic differentiation:

6-42) $\boxed{f(x) = \dfrac{(3x^4+x^2)^6}{(1+x+x^2)^8}}$

$$\ln f(x) = 6\left[\ln(3x^4+x^2)\right] - 8\left[\ln(1+x+x^2)\right]$$

$$\frac{f'(x)}{f(x)} = 6 \cdot \frac{(12x^3+2x)}{3x^4+x^2} - 8 \cdot \frac{(1+2x)}{(1+x+x^2)}$$

$$\Rightarrow f'(x) = (f(x))\left[\frac{6(12x^3+2x)}{3x^4+x^2} - \frac{8(1+2x)}{(1+x+x^2)}\right]$$

$$\boxed{f'(x) = \frac{(3x^4+x^2)^6}{(1+x+x^2)^8} \cdot \left[\frac{6(12x^3+2x)}{(3x^4+x^2)} - \frac{8(1+2x)}{(1+x+x^2)}\right]}$$

6-43) $f(x) = (\ln x)^x \Rightarrow \ln f(x) = \ln\left[(\ln x)^x\right]$

$$\Rightarrow \ln f(x) = [x] \cdot [\ln(\ln x)]$$

$$\Rightarrow \frac{f'(x)}{f(x)} = (1)\cdot(\ln(\ln x)) + (\cancel{x}) \cdot \left(\frac{1}{\ln x} \cdot \frac{1}{\cancel{x}}\right)$$

log. diff. $\left.\right\}$ $\boxed{f'(x) = (\ln x)^x \left[\ln(\ln x) + \frac{1}{\ln x}\right]}$

6-60) $f(x) = \ln\left(\frac{x \cdot e^x}{1+x^2}\right)$, $a=2$

$\boxed{\text{Find } f'(2) = ?}$ [find the tangent line eqn. to $f(x)$ at the point where $x_0 = -1$].

$f(x) = \ln(x \cdot e^x) - \ln(1+x^2)$

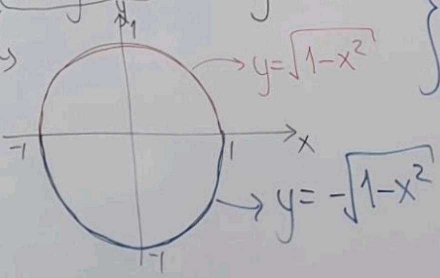
$$f'(x) = \frac{1 \cdot e^x + x \cdot e^x}{x \cdot e^x} - \frac{2x}{1+x^2}$$

$$f'(2) = \frac{e^2 + 2e^2}{2 \cdot e^2} - \frac{2(2)}{1+(2)^2} = \boxed{\frac{3}{2}e^2 - \frac{4}{5}}$$

Implicit Differentiation:

An equation involving x & y may define y as a function of x. This is called an <u>implicit function</u>. For example, the following equations define y implicitly as a function of x:

* $\boxed{x^2+y^2=1} \Rightarrow y=\mp\sqrt{1-x^2}$ explicit functions of x



* $ye^y+2x-\ln y=0$
* $3xy+x^2y^3+x=5$
* $e^x+e^y=\sqrt{x+2y}$

without solving y as a function of x we can still find $y'(x)$ [having assumed $y=y(x)$] by using $\boxed{\text{implicit differentiation}}$

The following equations define y explicitly:

* $y=x^3-5x^2$
* $y=\ln(x^2-e^x)$
* $y=x^3+\sqrt{x}+xe^x$
* $y=\dfrac{1}{1+e^{x^2-x}}$

y' can be found as a function of x using the differentiation formulas we have learned so far!

The main idea of implicit differentiation is:

* differentiate every term w.r.t. x
* Solve for $y'(x)$.

Ex. 7-1: Find the slope (hence tg. line eqn.) to the curve: $x^2+y^2=4$ at the point $(1,\sqrt{3})$.

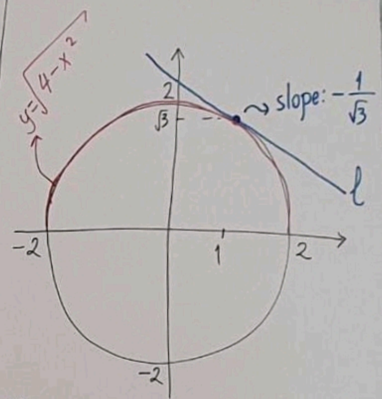
Diff. $x^2+y^2=4$ w.r.t. "x" assuming $y=y(x)$:

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(4)$$

$$2x+(2y)\left(\frac{dy}{dx}\right)=0$$

$$\cancel{2}y\cdot y'=-\cancel{2}x \Rightarrow \boxed{y'=-\frac{x}{y}}$$

at $(1,\sqrt{3})$: $\left. y'\right|_{(1,\sqrt{3})}=-\frac{1}{\sqrt{3}}$



ℓ: tangent line to the given circle at $(1,\sqrt{3})$ is:

$$y-\sqrt{3}=\left(-\frac{1}{\sqrt{3}}\right)(x-1)$$

$$y=-\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}+\sqrt{3}=-\frac{1}{\sqrt{3}}x+\frac{4}{\sqrt{3}}$$

tg. line: $\boxed{y=-\frac{1}{\sqrt{3}}x+\frac{4}{\sqrt{3}}}$

Ex.7.2: Find the slope of the tangent line to the curve

$y'(1) = ?$

$$x^8 + 4x^2y^2 + y^8 = 6 \quad \text{at the pt. } (1,1)$$

$$\frac{d}{dx}(x^8) + 4\frac{d}{dx}(x^2y^2) + \frac{d}{dx}(y^8) = \frac{d}{dx}(6)$$

$$8x^7 + 4[2x \cdot y^2 + x^2(2y \cdot y')] + [8y^7 \cdot y'] = 0$$

$$y'[8x^2y + 8y^7] = -8x^7 - 8xy^2$$

$$\Rightarrow \boxed{y'(x) = -\frac{8x^7 + 8xy^2}{8y^7 + 8x^2y}}$$

$$y'\Big|_{(1,1)} = -\frac{8(1)^7 + 8(1)(1)^2}{8(1)^7 + 8(1)^2(1)} = -1$$

tg. line eqn.: at (1,1) $\quad y - 1 = (-1)(x-1) = -x + 1 \Rightarrow \boxed{y = -x + 2}$

Ex.7.3: Find y' at $(0,0)$ where

$$(1 + x + 2y)(e^y) + 3xe^x = 1 + x^2 + y^2$$

Soln.: Using implicit diff. w.r.t. x:

$$(1 + 2y')(e^y) + (1 + x + 2y)(e^y y') + 3[(1)e^x + x \cdot e^x] = 2x + 2y \cdot y'$$

$$y'[2e^y + e^y(1 + x + 2y) - 2y] = -e^y - 3e^x(1+x) + 2x$$

$$y' = -\frac{e^y + 3e^x(1+x) - 2x}{2e^y + e^y(1 + x + 2y) - 2y}$$

$$y'\Big|_{(0,0)} = -\frac{e^0 + 3e^0(1+0) - 2(0)}{2e^0 + e^0(1 + 0 + 2(0)) - 2(0)} = -\frac{1+3}{2+1} = \boxed{-\frac{4}{3}}$$

tg. line eqn.:

$$y - 0 = -\frac{4}{3}(x - 0)$$

$$y = -\frac{4}{3}x$$

$$\boxed{4x + 3y = 0}$$