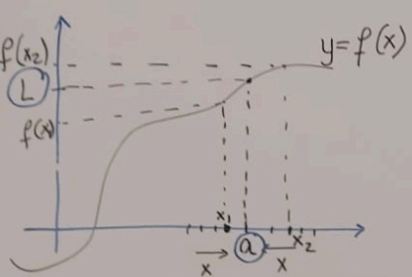
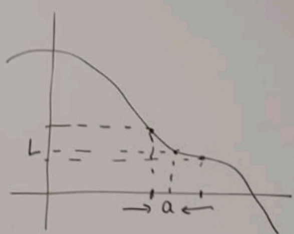
Limits:





If the function f approaches "L" as x approaches "a" we say that the limit of f at "a" is L, and we write:

$$\boxed{\lim_{x \to a} f(x) = L}$$

One-sided limits:

* If x approaches a with values larger than a, we write: $x \to a^+$ $(x > a)$

* If x " " " " smaller " ", we write: $x \to a^-$ $(x < a)$

* If f approaches L_1 as $x \to a^+$ $(x > a)$, we say that f has a <u>right-limit</u> and write $\lim_{x \to a^+} f(x) = L_1$ (<u>right-hand limit of f</u>)

* If f approaches L_2 as $x \to a^-$ $(x < a)$, we say that f has a <u>left-limit</u> and write $\lim_{x \to a^-} f(x) = L_2$ (<u>left-hand limit of f</u>)

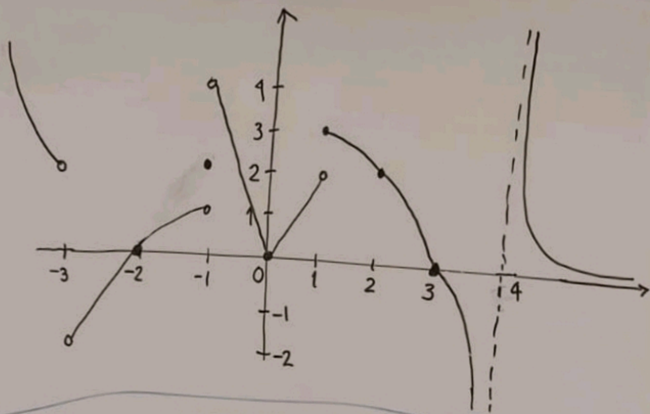
*We say that f has a limit L at $x=a \Leftrightarrow$

$$\boxed{\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)}$$

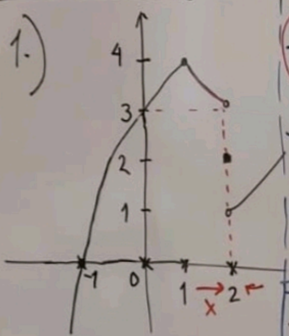
Estimating a limit from a Graph:

Ex: Find the following limits and their values using the graph of $f(x)$:

2.)



1.)



$f(-1)=0$, $\lim\limits_{\substack{x\to -1^- \\ (x<-1)}} f(x) = 0$, $\lim\limits_{\substack{x\to -1^+ \\ (x>-1)}} f(x) = 0 \Rightarrow \boxed{\lim\limits_{x\to -1} f(x) = 0}$

$\boxed{f(0)=3}$, $\lim\limits_{\substack{x\to 0^- \\ (x<0)}} f(x) = 3$, $\lim\limits_{\substack{x\to 0^+ \\ (x>0)}} f(x) = 3 \Rightarrow \boxed{\lim\limits_{x\to 0} f(x) = 3}$

$\boxed{f(1)=4}$, $\lim\limits_{\substack{x\to 1^- \\ (x<1)}} f(x) = 4$, $\lim\limits_{\substack{x\to 1^+ \\ (x>1)}} f(x) = 4 \Rightarrow \boxed{\lim\limits_{x\to 1} f(x) = 4}$

$f(2)=2$,

$\lim\limits_{\substack{x\to 2^- \\ (x<2)}} f(x) = 3$ (left-limit)

$\neq$

$\lim\limits_{\substack{x\to 2^+ \\ (x>2)}} f(x) = 1$ (right-limit)

$\Rightarrow$ $\lim\limits_{x\to 2} f(x) =$ D.N.E. (does not exist)

$f(-3) =$ not defined (undefined) ; $\lim\limits_{\substack{x \to -3^- \\ (x<-3)}} f(x) = 2$, $\lim\limits_{\substack{x \to -3^+ \\ (x>-3)}} f(x) = -2 \Rightarrow \lim\limits_{x \to -3} f(x) =$ d.n.e.

$f(-2) = 0$, $\lim\limits_{\substack{x \to -2^- \\ (x<-2)}} f(x) = 0$, $\lim\limits_{\substack{x \to -2^+ \\ (x>-2)}} f(x) = 0 \Rightarrow \lim\limits_{x \to -2} f(x) = 0$

$f(-1) = 2$, $\lim\limits_{\substack{x \to -1^- \\ (x<-1)}} f(x) = 1$, $\lim\limits_{\substack{x \to -1^+ \\ (x>-1)}} f(x) = 4 \Rightarrow \lim\limits_{x \to -1} f(x) =$ d.n.e.

$f(0) = 0$, $\lim\limits_{\substack{x \to 0^- \\ (x<0)}} f(x) = 0$, $\lim\limits_{\substack{x \to 0^+ \\ (x>0)}} f(x) = 0 \Rightarrow \lim\limits_{x \to 0} f(x) = 0$

$f(1) = 3$, $\lim\limits_{\substack{x \to 1^- \\ (x<1)}} f(x) = 2$, $\lim\limits_{\substack{x \to 1^+ \\ (x>1)}} f(x) = 3 \Rightarrow \lim\limits_{x \to 1} f(x) =$ d.n.e.

$f(2) = 2 = \lim\limits_{x \to 2} f(x)$

$f(3) = 0 = \lim\limits_{x \to 3} f(x)$

$f(4) =$ undefined ; $\lim\limits_{\substack{x \to 4^- \\ (x<4)}} f(x) = -\infty$

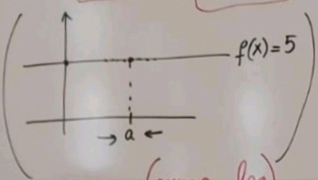
$\lim\limits_{\substack{x \to 4^+ \\ (x>4)}} f(x) = \infty \; (+\infty)$

$\Big\}\; \lim\limits_{x \to 4} f(x) =$ d.n.e.

Note: if either left or right limit at $x=a$ does not exist (i.e. $\mp\infty$) $\Rightarrow$ then we say that the limit d.n.e. at $x=a$

Properties of Limit:

1-) If $f(x)=c \Rightarrow \boxed{\lim_{x\to a} f(x)} = \lim_{x\to a} c = \boxed{c}$



2-) If f is a polynomial function (expon., log)

then $\boxed{\lim_{x\to a} f(x) = f(a)}$

3-) $\lim_{x\to a} x^n = a^n$, for any positiver integer "n".

Suppose that $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$

$b\neq 0$

$\frac{0}{b}=0$, $\frac{b}{0}$ = d.n.e. (undefined)

4-) $\lim_{x\to a}\left(f(x)\mp g(x)\right) = L \mp M$

5-) $\lim_{x\to a}\left[f(x)\cdot g(x)\right] = L\cdot M$

6-) $\lim_{x\to a}\left(c\, f(x)\right) = c\cdot L$ (for any constant "c")

7-) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$ (provided $M\neq 0$)

8-) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$(a\mp b)^2=a^2\mp 2ab+b^2$

$(a\mp b)^3=a^3\mp 3a^2b+3ab^2\mp b^3$

Examples: Evaluate the following limits, if they exist.

1-) $\lim_{x\to 1}(x^2+x+1)=(1)^2+(1)+1=3$

2-) $\lim_{x\to -3}\left(\frac{x-3}{x+5}\right)=\frac{-3-3}{-3+5}=\frac{-6}{2}=-3$

3-) $\lim_{x\to 0}\left(\frac{x}{x^3-4x+3}\right)=\frac{0}{0^3-4(0)+3}=\frac{0}{3}=\boxed{0}$

4-) $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right) \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 1}\left[\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}\right] = 1+1 = 2$ $(x\neq 1)$

5-) $\lim_{x\to 1}\left(\frac{x^3-1}{x-1}\right) \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 1}\left[\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}\right]$ $(x\neq 1)$

$= \lim_{x\to 1}(x^2+x+1) = 3$

6.) $\lim\limits_{x\to 2}\left(\dfrac{x^2-5x+6}{x^2-3x+2}\right) \underset{\downarrow \left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 2} \dfrac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x-1)} = \dfrac{2-3}{2-1} = \dfrac{-1}{1} = -1$

$x \neq 2$

7.) $\lim\limits_{x\to 3}\left[\dfrac{(x-3)}{x^2-9}\right] = \dfrac{1}{6}$ → HW

8.) $\lim\limits_{x\to -1}\sqrt{2x^2+3} = \sqrt{2(-1)^2+3} = \sqrt{5}$

9.) $\lim\limits_{x\to 4}\left(\dfrac{\sqrt{x}-1}{x+1}\right) = \dfrac{\sqrt{9}-1}{4+1} = \dfrac{1}{5}$

10.) $\lim\limits_{x\to 1}\dfrac{(\sqrt{x}-1)}{x-1} \underset{\downarrow \left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 1}\left[\dfrac{\cancel{(\sqrt{x}-1)}}{\cancel{(\sqrt{x}-1)}(\sqrt{x}+1)}\right] = \dfrac{1}{2}$

$x-1 = (\sqrt{x}-1)(\sqrt{x}+1)$

$x \neq 1$

11.) $\lim\limits_{x\to e} \ln(4x^2) = \ln(4(e^2)) = \ln(4e^2) = \ln 4 + \underbrace{\ln e^2}_{2} = \boxed{2+\ln 4}$

12.) $\lim\limits_{x\to 4} \log_2\left[\left(\dfrac{x}{x-2}\right)^2\right] = \log_2\left(\dfrac{4}{4-2}\right)^2 = \log_2 4 = 2\underbrace{\log_2 2}_{1}$

$= 2$

13.) $\lim\limits_{x\to 2}\left(e^x+2^x+4x\right) = e^2+2^2+4(2) = \boxed{12+e^2}$

Ex. 4-1: Evaluate $\boxed{\lim\limits_{x\to 2}\left(\dfrac{3}{x-2}\right)}$ d.n.e. (if it exists)

i) $\lim\limits_{x\to 2^-}\left(\dfrac{3}{x-2}\right) = \dfrac{3}{0^-} = -\infty$

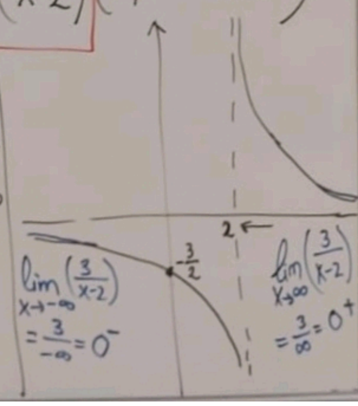
$(x<2)$

$(x-2<0)$

ii) $\lim\limits_{x\to 2^+}\left(\dfrac{3}{x-2}\right) = \dfrac{3}{0^+} = \infty$

$(x>2)$

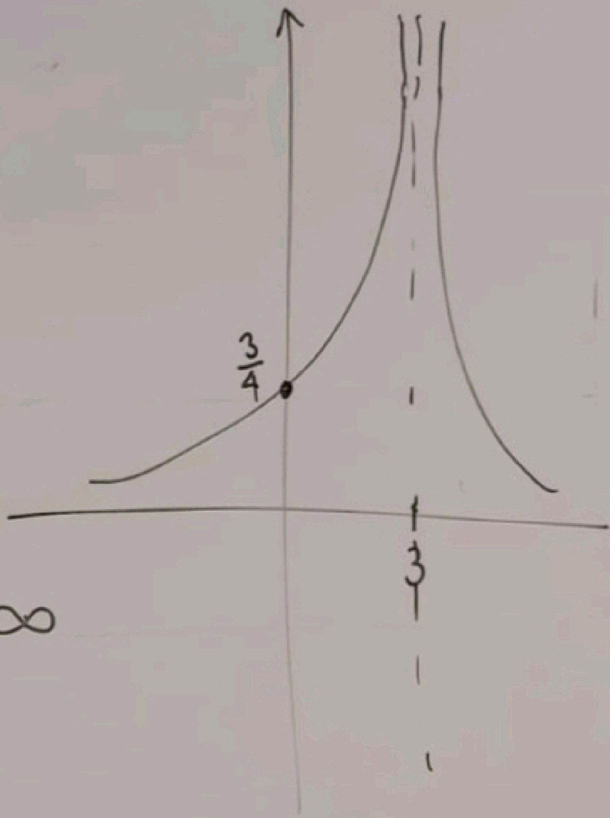
$(x-2>0)$



eg. $\lim_{x \to 2} \frac{3}{(x-2)^2} = \text{d.n.e.}$

$$\lim_{x \to 2^-} \frac{3}{(x-2)^2} = \frac{3}{(0^-)^2} = \frac{3}{0} = \infty$$

$$\left(\begin{array}{c} x<2 \\ x-2<0 \end{array} \right)$$



$$\lim_{x \to 2^+} \frac{3}{(x-2)^2} = \frac{3}{(0^+)^2} = \frac{3}{0} = \infty$$

$$\left(\begin{array}{c} x>2 \\ (x-2)>0 \end{array} \right)$$