Derivative:

Definition and Notation:

The derivative of the function $f(x)$ is the function $f'(x)$ defined by

$$\boxed{f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)}$$

definition of the derivative

or equivalently;

$$f'(x)=\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)$$

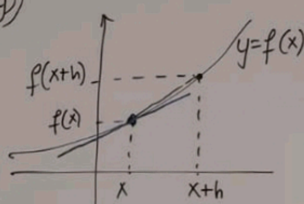
if the limits exist.

We think of the derivative as;

* the rate of change of a function f, or
* the slope of the curve $y=f(x)$ (at (x,y))

We will use y', $f'(x)$, $\frac{dy}{dx}$, $\frac{d}{dx}f(x)$ to denote derivatives and

$f'(a)$, $\left.\frac{dy}{dx}\right|_{x=a}$ to denote their values at a certain point.



$$\frac{f(x+h)-f(x)}{(x+h)-x}$$

* Note that; the derivative is a function, its value at a point is a number.

Differentiation Rules:

Product rule:

If f and g are differentiable at x, then $f \cdot g$ is differentiable at x and;

$$\frac{d}{dx}(f \cdot g) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$$

$$\left((f \cdot g)' = f' \cdot g + f \cdot g'\right)$$

Ex 6-4: Find $f'(x)$ for $f(x) = (x^4 + 14x)(7x^3 + 17)$

$$f'(x) = (4x^3 + 14) \cdot (7x^3 + 17) + (x^4 + 14x) \cdot (21x^2)$$

$$= 28x^6 + 68x^3 + 91x^3 + 238 + 21x^6 + 294x^3$$

$$= \boxed{49x^6 + 453x^3 + 238}$$

Quotient Rule:

If f & g are differentiable at x, and if $g(x) \neq 0$

$\Rightarrow \frac{f}{g}$ is differentiable at x and:

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{g^2}}$$

Reciprocal Rule:

$$\boxed{\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}}$$

Ex. 6.5: $f(x) = \frac{1}{x^n} \Rightarrow f'(x) = ?$

$$\Rightarrow f'(x) = \frac{-nx^{n-1}}{x^{2n}} = -n\,x^{n-1-2n} = -n\,x^{-1-n} = \frac{-n}{x^{n+1}}$$

* If f is a function and c is a constant $\Rightarrow$

$$\frac{d}{dx}(cf) = (cf)' = cf'$$

* $(f \mp g)' = f' \mp g'$

Ex 6.1: $\boxed{f(x) = 7x^3 - 18x}$

$$f'(x) = 21x^2 - 18 \Rightarrow f''(x) = 42x - 0 = 42x$$

$$\Rightarrow f'''(x) = 42 \Rightarrow f^{(IV)}(x) = 0$$

$$\Rightarrow f^{(n)}(x) = 0 \quad \forall n \geq 4$$

Ex 6.2: Find $f'(x)$ for $f(x) = \frac{7x^3 - 18x}{x} = \frac{7x^3}{x} - \frac{18x}{x}$

$$\Rightarrow \boxed{f(x) = 7x^2 - 18}$$

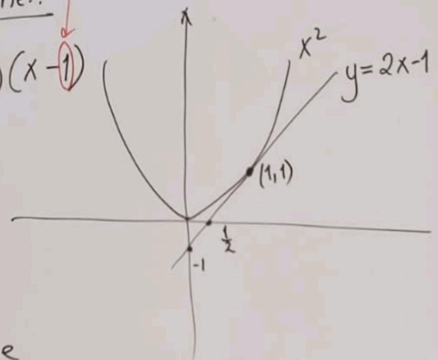
$$\boxed{f'(x) = 14x}$$

Ex 6.3: Find the equation of the tangent line to the graph of $f(x) = x^2$ at the point $(1, 1)$.

Soln.: slope $= f'(1) = 2$

$\to f'(x) = 2x \Rightarrow f'(1) = 2$

tang. line: $y - 1 = 2(x - 1)$



$$y - 1 = 2x - 2$$

$\boxed{y = 2x - 1}$ eqn. of tg. line to the graph of $y = x^2$ at $(1,1)$

Higher-order derivatives:

y'', $f''(x)$, $\frac{d^2y}{dx^2}$ } second derivative of $f(x)$

Third der.: $f'''(x)$, $\frac{d^3y}{dx^3}$, ...

n^{th} order deriv.: $\frac{d^ny}{dx^n}$

Differentiation formulas:

Using the defn. of derivative, we obtain:

* derivative of a constant: $\boxed{\frac{dc}{dx}=0}$ ($\underbrace{\text{constant}}_{y=f(x)=c}$)

* derivative of $f(x)=1 \Rightarrow \frac{d}{dx}(1)=0$

* deriv. of $f(x)=x$: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)-x}{h}$

$f(x+h)=x+h$

$= \lim_{h\to 0}\left(\frac{h}{h}\right) = 1$

$\Rightarrow \boxed{\frac{d}{dx}(x)=1}$ i.e. $\frac{d}{dx}(x^1)=1x^0=1\cdot 1=1$

* deriv. of x^n is: $\frac{d}{dx}(x^n)=nx^{n-1}$ $(n\in\mathbb{R})$

$$\frac{d}{dx}(x^n)=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}=\lim_{h\to 0}\frac{\cancel{x^n}+nx^{n-1}\cdot h+n(n-1)x^{n-2}\cdot h^2+\dots+h^n-\cancel{x^n}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}[nx^{n-1}+\underbrace{hn(n-1)x^{n-2}+\dots\dots+h^n}_{\text{terms involving } h}]}{\cancel{h}}=nx^{n-1}+0+\dots+0$$

$$\boxed{\frac{d}{dx}(x^n)=nx^{n-1}}$$ power rule

* deriv. of $f(x)=\sqrt{x}$ is:

$$\boxed{\frac{d}{dx}(x^{1/2})}=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2}x^{-1/2}=\boxed{\frac{1}{2\sqrt{x}}}$$

$$f'(x)=\frac{\left(\frac{3}{2}x^{1/2}-\frac{1}{2}x^{-3/2}\right)\left(x^{1/2}+x^{-1/2}\right)-\left(\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{-3/2}\right)\left(x^{3/2}+x^{-1/2}\right)}{\left(x^{1/2}+x^{-1/2}\right)^2}$$

$$=\frac{\frac{3}{2}x+\frac{3}{2}-\frac{1}{2}x^{-1}-\cancel{\frac{1}{2}x^{-2}}-\frac{1}{2}x^{2}-\frac{1}{2}x^{-1}+\frac{1}{2}(1)+\cancel{\frac{1}{2}x^{-2}}}{\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2=\left(x+2+\frac{1}{x}\right)}$$

HW: simplify the above result

or $$f(x)=\frac{x^{3/2}+\frac{1}{\sqrt{x}}}{\sqrt{x}+\frac{1}{\sqrt{x}}}=\frac{\frac{x^2+1}{\sqrt{x}}}{\frac{x+1}{\sqrt{x}}}=\frac{x^2+1}{\cancel{\sqrt{x}}}\cdot\frac{\cancel{\sqrt{x}}}{x+1}$$

$$\Rightarrow f(x)=\frac{x^2+1}{x+1}$$

$$\Rightarrow f'(x)=\frac{(2x)(x+1)-(1)(x^2+1)}{(x+1)^2}=\boxed{\frac{x^2+2x-1}{(x+1)^2}}$$

6.14) $f(x)=x^2\ln(x^3)=\boxed{3x^2\ln x}\Rightarrow f'(x)=3\left[(2x)(\ln x)+x^2\cdot\frac{1}{x}\right]=\boxed{6x\ln x+3x}$

or $=\boxed{2x\ln x^3+3x}$

6.17) $f(x)=\frac{1}{1+2x+3e^x}\Rightarrow f'=\frac{-(2+3e^x)}{(1+2x+3e^x)^2}$

6.19) $f(x)=(x^4)(e^x\cdot\ln x)\Rightarrow f'(x)=(4x^3)(e^x\ln x)+(x^4)\left(e^x\ln x+e^x\cdot\frac{1}{x}\right)$

$f'(x)=\boxed{(4x^3)(e^x\ln x)+x^4e^x\ln x+x^3e^x}$

6.51) $f(x)=\frac{\ln x}{x^4},\ a=1\Rightarrow f'(a)=?$ $\quad f'(1)=1-4(1)(\ln 1)=1$

$f'(x)=\frac{\left(\frac{1}{x}\right)(x^4)-(4x^3)(\ln x)}{(x^4)^2}=\frac{x^3-4x^3\ln x}{x^8}=x^{-5}-4x^{-5}\ln x$

6.52) $f(x)=(1+2x)e^x,\ f'(0)=?$

$f'(x)=(2)e^x+(1+2x)e^x=3e^x+2xe^x\Rightarrow f'(0)=3e^0+0=\boxed{3}$

Ex 6.6: $f(x) = \dfrac{1}{8x^2+12x+1}$

$$f'(x) = \frac{-(16x+12)}{(8x^2+12x+1)^2}$$

Ex. 6.7: $f(x) = \dfrac{2x+3}{5x^2+7}$

$$f'(x) = \frac{(2)(5x^2+7)-(10x)(2x+3)}{(5x^2+7)^2}$$

$$= \frac{10x^2+14-20x^2-30x}{(5x^2+7)^2}$$

$$= -\frac{10x^2+30x-14}{(5x^2+7)^2}$$

Deriv. of Exponentials & Logarithms:

$$\frac{d}{dx}(e^x) = e^x, \quad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln x) = \boxed{\frac{1}{\ln a \cdot x}}$$

Exercises: Evaluate the derivatives of the following functions:

6-7) $f(x) = \dfrac{x^3 - x}{\sqrt{x}} = x^{3-\frac{1}{2}} - x^{1-\frac{1}{2}} = x^{\frac{5}{2}} - x^{1/2}$

$$f'(x) = \frac{5}{2}x^{\frac{5}{2}-1} - \frac{1}{2}x^{\frac{1}{2}-1} = \boxed{\frac{5}{2}\overset{x^{3/2}}{\sqrt{x^3}} - \frac{1}{2\sqrt{x}}} = \boxed{\frac{5\sqrt{x^5}-1}{2\sqrt{x}}}$$

6-12) $f(x) = \dfrac{x^{3/2} + x^{-1/2}}{\sqrt{x} + \frac{1}{\sqrt{x}}} = \dfrac{x^{3/2} + x^{-1/2}}{x^{1/2} + x^{-1/2}}$