4.17) $\underbrace{\lim_{x\to\infty}\left(\frac{8e^{x}}{4+5e^{x}}\right)}_{=} = \frac{\infty}{\infty}$

$$\lim_{x\to\infty}\left[\frac{\cancel{e^{x}}(8)}{\cancel{e^{x}}\left[\frac{4}{e^{x}}+5\right]}\right] = \frac{8}{\underbrace{\frac{4}{\infty}}_{0}+5} = \boxed{\frac{8}{5}}$$

4.18) $\lim_{x\to\infty}\left(\sqrt{x^2+4x}-\sqrt{x^2-10x+1}\right) = \infty-\infty$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+4x}-\sqrt{x^2-10x+1}\right)\left(\sqrt{x^2+4x}+\sqrt{x^2-10x+1}\right)}{\left(\sqrt{x^2+4x}+\sqrt{x^2-10x+1}\right)}$$

$$=\lim_{x\to\infty}\left[\frac{(\cancel{x^2}+4x)-(\cancel{x^2}-10x+1)}{\sqrt{x^2+4x}+\sqrt{x^2-10x+1}}\right]=\lim_{x\to\infty}\left[\frac{14x-1}{\sqrt{x^2+4x}+\sqrt{x^2-10x+1}}\right]$$

$\sqrt{x^2}=|x|$

$x>0: \sqrt{x^2}=x$

$x<0: \sqrt{x^2}=-x$

$$\sqrt{x^2+4x}+\sqrt{x^2-10x+1}=\sqrt{x^2\left(1+\frac{4}{x}\right)}+\sqrt{x^2\left(1-\frac{10}{x}+\frac{1}{x^2}\right)}=\left\{(-x)\sqrt{}+\sqrt{} \text{ if } x\to-\infty\right\}$$

$$\lim_{x\to\infty}\left(\qquad\right)=\lim_{x\to\infty}\left[x\sqrt{1+\frac{4}{x}}+x\sqrt{1-\frac{10}{x}+\frac{1}{x^2}}\right]$$

$$=\lim_{x\to\infty}\frac{\cancel{x}\left(14-\frac{1}{x}\right)}{\cancel{x}\left[\sqrt{1+\frac{4}{x}}+\sqrt{1-\frac{10}{x}+\frac{1}{x^2}}\right]}$$

$$=\frac{14-0}{\sqrt{1+0}+\sqrt{1-0+0}}=\frac{14}{2}=\boxed{7}$$

Modify the problem as:

$$* \lim_{x \to -\infty} \left(\sqrt{x^2+4x} - \sqrt{x^2-10x+1} \right) = \sqrt{(-\infty)^2} - \sqrt{(-\infty)^2} = \infty - \infty$$

$$= \lim_{x \to -\infty} \frac{14x - 1}{(-x)\left\{ \sqrt{1+\frac{4}{x}} + \sqrt{1-\frac{10}{x}+\frac{1}{x^2}} \right\}}$$

$$= \lim_{x \to -\infty} \frac{\cancel{x}\left[14 - \frac{1}{x}\right]}{(-\cancel{x})\left\{ \sqrt{1+\frac{4}{x}} + \sqrt{1-\frac{10}{x}+\frac{1}{x^2}} \right\}}$$

$$= \frac{14-0}{-\left\{ \sqrt{1+0} + \sqrt{1-0+0} \right\}} = \frac{14}{-2} = -7$$

4-21) $$\lim_{x \to \infty} \frac{1}{2x - \sqrt{4x^2-5x+6}} = \frac{1}{\infty - \infty}$$

↳ inderterminate form

$$= \lim_{x \to \infty} \frac{2x + \sqrt{4x^2-5x+6}}{(2x)^2 - \left(\sqrt{4x^2-5x+6}\right)^2}$$

$$= \lim_{x \to \infty} \frac{2x + \overbrace{\sqrt{x^2\left(4 - \frac{5}{x} + \frac{6}{x^2}\right)}}^{x\sqrt{4-\frac{5}{x}+\frac{6}{x^2}}}}{\cancel{4x^2} - \left(\cancel{4x^2} - 5x + 6\right)}$$

$$= \lim_{x \to \infty} \frac{\cancel{x}\left(2 + \sqrt{4 - \frac{5}{x} + \frac{6}{x^2}}\right)}{\cancel{x}\left(5 - \frac{6}{x}\right)}$$

$$= \frac{2 + \sqrt{4 - 0 + 0}}{5 - 0} = \frac{4}{5}$$

4.22) $\lim_{x\to\infty}\left(\sqrt{2x^2-1}-\sqrt{x^2+1}\right)=\infty-\infty$

$$=\lim_{x\to\infty}\frac{(2x^2-1)-(x^2+1)}{\underbrace{\sqrt{2x^2-1}}_{\left(x\sqrt{2-\frac{1}{x^2}}\right)}+\underbrace{\sqrt{x^2+1}}_{\left(x\sqrt{1+\frac{1}{x^2}}\right)}}=\lim_{x\to\infty}\frac{x^{\cancel{2}}\left(1-\frac{2}{x^2}\right)}{\cancel{x}\left(\sqrt{2-\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}\right)}$$

$$=\frac{\infty(1-0)}{\sqrt{2-0}+\sqrt{1+0}}=\frac{\infty}{\sqrt{2}+1}=\infty$$

Ex. 5-5: Evaluate the limit, if it exists.
(Do not use L'Hôspital's rule)

$$\lim_{x\to 8^+}\left(\frac{x^2-10x+16}{\sqrt{x-8}}\right)\underset{\frac{0}{0}}{=}$$

$$=\lim_{x\to 8^+}\frac{(x-8)(x-2)}{\sqrt{x-8}}=\lim_{x\to 8^+}\frac{\left(\sqrt{x-8}\right)^{\cancel{2}}(x-2)}{\cancel{\left(\sqrt{x-8}\right)}}=\lim_{x\to 8^+}\left[\left(\sqrt{x-8}\right)(x-2)\right]$$

$$=0\cdot 6=\boxed{0}$$

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$: indeterminate forms

$$\lim_{x\to\mp\infty}\left(\frac{1}{x^p}\right)=0\quad(p>0)$$

Continuity:

We say that f is continuous at "a" if

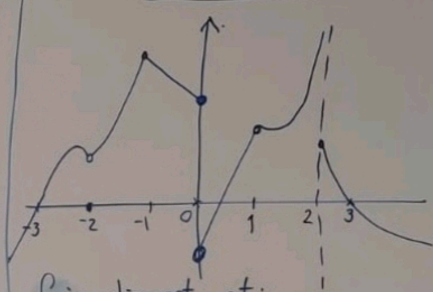
$$\boxed{\lim_{x\to a} f(x) = f(a)}$$

i.e.

* f must be defined at $x=a$
* $\lim_{x\to a} f(x)$ must exist
* $\lim_{x\to a} f(x)$ must equal $f(a)$.

Ex 5-6: Determine the points where $f(x)$ is discontinuous:

(Explain why)



f is discont. at:

* $x=-2$: $f(-2)$ and $\lim_{x\to 2} f(x)$ are not equal
* $x=0$: $\lim_{x\to 0} f(x)$: d.n.e. (right and left limits are not equal)
* $x=1$: $f(1)$ is not defined
* $x=2$: limit does not exist $\left(\lim_{x\to 2^-} f(x) = \infty\right)$

Ex. 5-7:

$$f(x)=\begin{cases} 2x^2+a & \text{if } x<2 \\ b & \text{if } x=2 \\ 3x-2 & \text{if } x>2 \end{cases}$$

Find "a" and "b" if $f(x)$ is continuous at $x=2$ (or everywhere).

* $f(2)=b$
* $\lim\limits_{x\to 2} f(x) = ?$

i) $\lim\limits_{\substack{x\to 2^- \\ (x<2)}} f(x) = \lim\limits_{x\to 2^-}(2x^2+a) = 2(2)^2+a = \boxed{8+a}$

ii) $\lim\limits_{\substack{x\to 2^+ \\ (x>2)}} f(x) = \lim\limits_{x\to 2^+}(3x-2) = 3(2)-2 = \boxed{4}$

left-limit, right-limit

$\lim\limits_{x\to 2} f(x)$ exists $\Leftrightarrow$ $8+a=4$

$\Rightarrow \boxed{a=-4}$

$$\Rightarrow \boxed{\lim_{x\to 2} f(x) = 4}$$

f is cont. at $x=2$ (hence everywhere) $\Leftrightarrow$

$b = f(2) = \lim\limits_{x\to 2} f(x) = 4$

$\Rightarrow \boxed{b=4}$

So for $\boxed{\begin{array}{l} a=-4 \\ b=4 \end{array}}$ f is cont. at $x=2$

Ex. 5-8: $f(x)=\begin{cases} \log\left(\frac{x}{2}+b\right) & \text{if } x<8 \\ x\left(\sqrt{x-8}+\frac{1}{4}\right) & \text{if } x\geq 8 \end{cases}$

Find "b" if f is continuous at $x=8$.

$* \ f(8) = 8\left(\sqrt{0}+\frac{1}{4}\right) = \boxed{2}$

$* \ \lim\limits_{x\to 8^-} f(x) = \lim\limits_{x\to 8^-} \log\left(\frac{x}{2}+b\right) = \log\left(\frac{8}{2}+b\right) = \log(4+b) = 2$

$\left(\lim\limits_{x\to 8} f(x) = f(8)\right.$.

$\Rightarrow \log_{10}(4+b) = 2 \Leftrightarrow 4+b = 10^2$

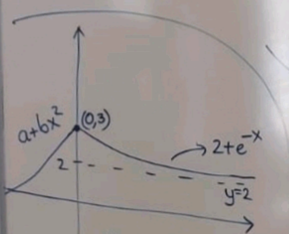
$\Rightarrow b = 100-4 = 96$

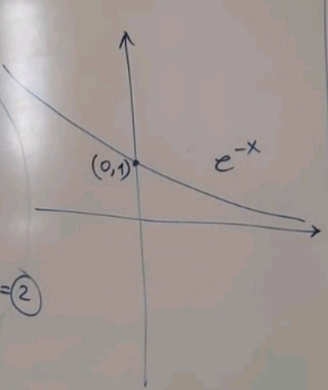
So for $b=96$ f is cont. at $x=8$

Domain: $\frac{x}{2}+96>0 \Rightarrow \frac{x}{2}>-96 \Rightarrow \boxed{x>-192}$

Find the values of constants that will make the following functions cont. everywhere:

5-21) $f(x)=\begin{cases} a+bx^2 & \text{if } x<0 \\ b & \text{if } x=0 \\ 2+e^{-x} & \text{if } x>0 \end{cases}$





$\lim_{x\to\infty}(2+e^{-x})=2+0=2$

* $f(0)=b$
* $\lim_{x\to 0} f(x)=?$

i) $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(2+e^{-x})=2+e^{-0}=2+1=3$
$(x>0)$

ii) $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(a+bx^2)=a$
$(x<0)$

in order for $f(x)$ to have a limit at $x=0$:

$\boxed{a=3}$

$\Rightarrow$ * for cont. at $x=0$: $\boxed{b=f(0)}=\boxed{\lim_{x\to 0} f(x)=3}$

$\Rightarrow \boxed{b=3}$

5-24) $f(x) = \begin{cases} e^{ax} & \text{if } x \leq 0 \\ \ln(b+x^2) & \text{if } x > 0 \end{cases}$

$* \; f(0) = e^{a(0)} = 1$

$* \; \lim_{x \to 0} f(x):$

i) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^{ax} = \boxed{1}$

in order for $f(x)$ to have a limit at $x=0$

ii) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \ln(b+x^2) = \boxed{\ln b}$

$\Rightarrow \ln b = 1 \Rightarrow b = e^1 = e$

$\Rightarrow \boxed{b=e}$

$f(0) = e^{a(0)} = 1$ for any $\boxed{a}$

Ans: $b=e$
a: arbitrary