ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 111 - Mathematics for Economics and Social Sciences I

2022-2023 Fall

FINAL EXAM

09.01.2023, 10:00

SOLUTIONS

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 100 minutes

Question	Grade	Out of
1		14
2		24
3		19
4		10
5		23
6		21
Total		111

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 questions.

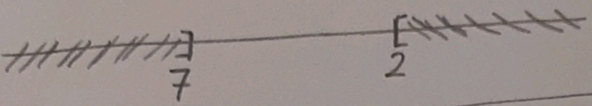
3) SHOW ALL YOUR WORK. No points will be given to correct answers without reasonable work.

1) a) (7 pts.) Find the solution set of the expression: $|2x+5|+1 \geq 10$.
Clearly indicate the solution set.

$|2x+5| \geq 9 \Rightarrow 2x+5 \geq 9$ or $-(2x+5) \geq 9$

$2x \geq 4$ $\quad\quad$ $-2x \geq 9+5 = 14$

$\boxed{x \geq 2}$ or $2x \leq -14 \Rightarrow \boxed{x \leq -7}$



$\boxed{\text{Solution set: } (-\infty, -7] \cup [2, \infty)}$

b) (7 pts.) Solve the equation: $\ln(\overset{>0}{x+1}) + \ln(\overset{>0}{x+7}) = \ln(\overset{>0}{2x-1})$.

$\ln[(x+1)(x+7)] = \ln(2x-1)$

$\Rightarrow x^2+8x+7 = 2x-1$

$\Rightarrow x^2+6x+8=0$

$(x+4)(x+2)=0$

$\boxed{x=-4 \text{ or } x=-2}$

$\ln(x+1)$ defined for $x>-1$
$\ln(x+7)$ // // $x>-7$
$\ln(2x-1)$ // // $x>\frac{1}{2}$

all the terms are defined when $x>\frac{1}{2}$

But $x>\frac{1}{2}$ should be satisfied for all the 3 $\ln(\cdots)$ terms to be defined $\Rightarrow$

So; $\boxed{\text{S.S.}: \{\emptyset\}}$

2) Evaluate the following limits if they exist. Express the solutions clearly. DO NOT USE L'HOPITAL'S RULE!

a) (8 pts.) Evaluate $\lim_{x\to\infty}\left(\frac{3}{x}-\frac{2x^2}{1+x^2}\right)$

$$= \frac{3}{\infty} - \lim_{x\to\infty}\left[\frac{2\cancel{x^2}}{\cancel{x^2}\left(\underbrace{\frac{1}{x^2}}_{0}+1\right)}\right] = 0-2 = \boxed{-2}$$

b) (8 pts.) $\lim_{x\to 2}\left(\frac{3x^2-x-10}{x^2+5x-14}\right) = \left(\frac{0}{0}\right)$

$$= \lim_{x\to 2}\left[\frac{(3x+5)\cancel{(x-2)}}{(x+7)\cancel{(x-2)}}\right] = \frac{6+5}{2+7} = \boxed{\frac{11}{9}}$$

c) (8 pts.) $\lim_{x\to 4}\left(\frac{x-4}{4-\sqrt{x+12}}\right) = \left(\frac{0}{0}\right)$

$$= \lim_{x\to 4} \frac{(x-4)(4+\sqrt{x+12})}{\underbrace{(4-\sqrt{x+12})(4+\sqrt{x+12})}_{16-(x+12)}} = \lim_{x\to 4}\frac{\cancel{(x-4)}(4+\sqrt{x+12})}{-\cancel{(x-4)}}$$

$$= -\left(4+\sqrt{4+12}\right) = \boxed{-8}$$

3) a) (10 pts.) Let the function

$$f(x) = (lnx)^x$$

be given. Find derivative of the function $\boxed{f(x) \text{ at } x = e}$ $(f'(e) = ?)$
(Hint: Use logarithmic differentiation)

$$f(x) = (\ln x)^x \Rightarrow \ln f(x) = \ln(\ln x)^x = x \cdot \ln(\ln x)$$

$$\Rightarrow \frac{f'}{f} = (1) \cdot [\ln(\ln x)] + (x)\left[\frac{1}{\ln x} \cdot \frac{1}{x}\right] = \left[\ln(\ln x) + \frac{1}{\ln x}\right]$$

$$\Rightarrow f'(x) = f(x) \cdot \left[\ln(\ln x) + \frac{1}{\ln x}\right] = (\ln x)^x \cdot \left[\ln(\ln x) + \frac{1}{\ln x}\right]$$

$$\Rightarrow \boxed{f'(e)} = \underbrace{(\ln e)}_{1}^{e} \cdot \left[\underbrace{\ln \underbrace{(\ln e)}_{1}}_{\ln 1 = 0} + \frac{1}{\ln e}\right] = 1 \cdot [0 + 1] = \boxed{1}$$

b) (9 pts.) Find y' using implicit differentiation, and simplify your answer as much as possible for $\boxed{x^3y + e^{xy^2} = 5.}$

$$\frac{d}{dx}: x^3y + e^{xy^2} = 5 \Rightarrow (3x^2 \cdot y + x^3 \cdot y') + e^{xy^2} \cdot (1 \cdot y^2 + x \cdot 2y \cdot y') = 0$$

$$\Rightarrow y'(x^3 + 2xy \cdot e^{xy^2}) = -3x^2y - y^2 \cdot e^{xy^2}$$

$$\Rightarrow \boxed{y' = -\frac{3x^2y + y^2e^{xy^2}}{x^3 + 2xye^{xy^2}}}$$

4) (10 pts.) Find real numbers a and b that make $f(x)$ continuous on $\mathbb{R}$ where

$$f(x) = \begin{cases} \dfrac{16}{x^2} + a, & \text{if } 2 < x, \\ 5, & \text{if } x = 2, \\ 3x - b, & \text{if } x < 2. \end{cases}$$

Explain your answer in detail.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(\frac{16}{x^2} + a \right) = \frac{16}{4} + a = \boxed{4 + a}$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - b) = 3(2) - b = \boxed{6 - b}$$

For continuity at $x = 2$:

$$\left. 4 + a = f(2) = 5 = 6 - b \right\} \Rightarrow \begin{array}{l} 4 + a = 5 \Rightarrow \boxed{a = 1} \\ 6 - b = 5 \Rightarrow \boxed{b = 1} \end{array}$$

* $f(x)$ is cont. for $x > 2$ and $x < 2$ since f is a polynomial for such x-values

$\Rightarrow$ f is cont. everywhere when $\boxed{a = 1, b = 1}$

5) Given that $\boxed{f(x) = e^{-x^2}}$

a) (6 pts.) Find the intervals where $f(x)$ is increasing and decreasing.

$$f'(x) = e^{-x^2} \cdot (-2x) = \begin{cases} 0 & \text{when } x=0 \\ \text{d.n.e., no such } x\text{-value} \end{cases}$$

$\Big\}$ $x=0$ is the only crit. pt. $\Rightarrow$ $\big((0,f(0))=(0,1)\big)$

x		0	
f'	$+$	0	$-$
f	$\nearrow$		$\searrow$

$$\boxed{\begin{array}{l} f \text{ increasing}: (-\infty, 0) \\ f \text{ decreasing}: (0, \infty) \end{array}}$$

b) (6 pts.) Find the local extrema of the function $f(x)$.

At $(0,1)$ there is a local maximum pt.

c) (6 pts.) Find the intervals where $f(x)$ is concave up and concave down.

$$f''(x) = -2\left[(1).e^{-x^2} + (x).e^{-x^2}.(-2x)\right] = -2e^{-x^2}\left[1-2x^2\right] = 0$$

$$\Rightarrow x = \mp\frac{1}{\sqrt{2}}$$

$\left(\mp\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$: inf. pts

x		$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

f concave up on: $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$

f concave down on: $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

d) (5 pts.) Find the absolute extrema of $f(x)$ on $[-1, 2]$.

	x	$f(x)$	
crit. pt. $\leftarrow$	0	1	$\longrightarrow$ max. value
end pts.	-1	$\frac{1}{e}$	
	2	$e^{-4} = \frac{1}{e^4}$	$\longrightarrow$ min. value

* f has abs. max. value of 1 at $x=0$.

* f has abs. min. value of $\frac{1}{e^4}$ at $x=2$.

6) Evaluate the following integrals.

a) (7 pts.) $\int_{-1}^{1} (3x^2 - 1)^7 5x\, dx = \int_{u=2}^{u=2} u^7 \cdot \frac{5}{6}\, du = \frac{5}{6} \frac{u^8}{8} \Big|_2^2 = \boxed{0}$

(upper limit = lower limit)

$\boxed{u = 3x^2 - 1}$

$du = 6x\, dx$

$\boxed{\frac{1}{6} du = x\, dx}$

$x = -1 \Rightarrow u = 3(-1)^2 - 1 = \boxed{2}$

$x = 1 \Rightarrow u = 3(1)^2 - 1 = \boxed{2}$

b) (7 pts.) $\int e^{5x^3} 7x^2\, dx = 7 \int e^u \cdot \frac{1}{15}\, du = \frac{7}{15} e^u + C$

$u = 5x^3$

$du = 15x^2 dx$

$\frac{1}{15} du = x^2 dx$

$= \boxed{\frac{7}{15} e^{5x^3} + C}$

c) (7 pts.) $\int_0^1 \frac{x + 2x^3}{1 + x^2 + x^4} dx = \int_1^3 \frac{\frac{1}{2} du}{u} = \frac{1}{2} \ln u \Big|_1^3$

$\boxed{u = 1 + x^2 + x^4}$

$du = (2x + 4x^3)\, dx$

$= 2(x + 2x^3) dx$

$\boxed{\frac{1}{2} du = (x + 2x^3) dx}$

$= \frac{1}{2} \ln 3 - \underbrace{\ln 1}_{=0}$

$= \boxed{\ln \sqrt{3}}$

$\boxed{\begin{array}{l} x = 0 \Rightarrow u = 1 \\ x = 1 \Rightarrow u = 3 \end{array}}$